ON THE MAXIMALITY OF CLASS OF RANK-2 DISTRIBUTIONS WITH 5-DIMENSIONAL CUBE

A Thesis

by

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ABSTRACT

This thesis is devoted to aspects of the local differential geometry of regular, bracket-generating distributions. The classical discrete invariants of such distributions at a point are its degree of nonholonomy and small growth vector, both of which encode how quickly the iterated Lie brackets of the distribution saturate the tangent space of the manifold at a point. Recently, Boris Doubrov and Igor Zelenko introduced a new discrete invariant of these distributions at a point called the Jacobi symbol, and constructed canonical frames for all distributions with given Jacobi symbol. Their constructions, however, require an additional generic assumption called the maximal class condition. This condition can be formulated in purely control-theoretic language as a property of the endpoint map along special curves called abnormal trajectories. There is a strong belief that the condition of maximal class is essentially redundant, which is to say that all bracket-generating distributions are of maximal class or become maximal class after a natural reduction process.

The aim of this thesis is to develop general tools for proving this conjecture and to apply these tools for the verification of the conjecture for a number of cases. We begin by proving that the maximal class condition is essentially determined by its Tanaka symbol, which is the graded space associated with the filtration of the distribution. We are most interested in proving that rank-2 distributions of dimension $n \geq 6$ and small growth vector $(2, 3, 5, \ldots)$ are of maximal class, and make our first steps towards this goal by proving that all $(2, n)$-distributions with Tanaka symbol isomorphic to the free truncated graded Lie algebra with 2 generators and degree of nilpotency 4 or 5 are of maximal class. We then provide calculations proving that all $(2, n)$-distributions associated with Monge ODEs are of maximal class.

DEDICATION

To Dennis, Chien, Delia, Cristin, and Sarah.

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1. INTRODUCTION

This work is motivated by, and makes a contribution to, an ongoing research program for reducing the local equivalence problem for vector distributions on a smooth manifold to the study of the differential geometry of curves of flags of symplectic subspaces. This research program, initiated by Andrei Agrachev and Igor Zelenko in [1, 2, 7, 8] and continued by Boris Doubrov and I. Zelenko in [16, 17, 32], is a novel approach to the solution of the local equivalence problem for vector distributions by means of a so-called *symplectification procedure* consisting of a lifting of the distribution to a special submanifold of the cotangent bundle foliated by special curves called *abnormal extremals.* The dynamics of this lift along an abnormal extremal is described by a special curve in a Grassmannian called the *Jacobi curve* of the abnormal extremal. Studying the differential geometry of these curves allows one to construct canonical frames for distributions according to certain discrete data given by the Jacobi curves of generic abnormal extremals. However, the construction of the canonical frame relies on a generic assumption called *maximal class* which, roughly speaking, means that the Jacobi curves of generic abnormal extremals fill out a maximal possible vector space. Distributions of non-maximal class have yet to be found and, as we discuss below, the primary objective of this work is to develop general tools for verifying the conjecture that *all regular, bracket-generating distributions of rank 2 and higher are of maximal class, or become maximal class through a natural reduction process.*

Recall that a rank-r distribution on a n-dimensional manifold M is a smooth subbundle of TM with r-dimensional fibers. The germs of distributions D_1, D_2 on M are said to be equivalent if there exists a local diffeomorphism $\varphi : U_1 \to U_2$

between neighborhoods U_1 of $q_1 \in M$ and U_2 of $q_2 = \varphi(q_1)$ such that $\varphi_* D_1 = D_2$.

The rank of the distribution and dimension of the manifold determine the number of functional parameters necessary for defining such a local diffeomorphism. If D_1 and D_2 are rank r distributions on an n-dimensional manifold then at least $r(n-r)-n$ many functional parameters are necessary [23]. This quantity is non-positive in only 3 cases: when $r = 1$ the distributions are line bundles and the study of their equivalence reduces to application of the Frobenius Theorem; when $r = n-1$ the distributions are called *contact* and their germs are described by *Darboux normal forms*; when $r = 2$ and $n = 4$ the distributions are of so-called *Engel form*. We refer to Chapter 6 of [24] for further details.

We expect functional parameters when $r(n-r)-n$ is positive, and the first such case $(r,n) = (2,5)$ was considered by Cartan in a famous paper [11]. His methodology for constructing a canonical frame, *i.e.* a frame on some bundle over M chosen in a canonical way, is known as the *Cartan equivalence method* or *Cartan prolongation method*.

Recall that the *filtration* of a distribution is the flag of subspaces obtained by iterated Lie brackets of vector fields in the distribution, such that the i^{th} subspace of the flag at $q \in M$ is given by $D^i(q) = D^{i-1}(q) + [D, D^{i-1}](q)$ for $i > 1$ and $D^1 = D$; see also our definition preceding equation (2.1) in Chapter 2. In [27, 28], Noboru Tanaka found a very natural algebraic interpretation of the Cartan prolongation method. First, he introduced the *symbol* of a distribution, which is a graded Lie algebra whose underlying vector space is the graded space associated with the filtration; we provide a complete definition in Section 2.3. Second, he defined an algebraic prolongation of the symbol that controls, in a sense, the procedure of constructing a canonical frame. However, the canonical frame construction in Tanaka theory depends on the symbol, which for a generic distribution can be non-isomorphic at different points

[31]. This necessitates the classification of all n-dimensional graded nilpotent Lie algebras with r generators in order to construct canonical frames for the equivalence problem using Tanaka theory and this classification becomes quite complicated even for low-dimensional distributions [21].

Recently, B. Doubrov and I. Zelenko introduced a new discrete invariant of bracket-generating distributions called the *Jacobi symbol*. In contrast to the Tanaka symbol the Jacobi symbol is locally constant in a neighborhood of a generic point and can be described by certain skew Young diagrams. In [32], Doubrov and Zelenko construct canonical frames for all distributions with given Jacobi symbol using the symplectification procedure which, as mentioned earlier, assumes the distribution satisfies the maximal class condition.

As mentioned above, the symplectification procedure is based on the construction of a curve of flags obtained by lifting the distribution to a linear symplectic space. Let $\pi : T^*M \to M$ be the canonical projection, D a rank-r bracket-generating distribution, and $\mathcal{J} := \pi^* D \subset TT^*M$ denote the lift or symplectification of D. This lift is a distribution with certain naturally-defined characteristic subdistributions and the symplectic procedure for the local equivalence problem takes judicious advantage of this geometric structure. First, the distribution is lifted at points in the special abnormal extremal curves of the cotangent bundle. Then, we recursively construct the flag by taking Lie brackets of each subspace of the flag with a distinguished subdistribution $\mathcal{C} \subset \mathcal{J}$; we describe this more precisely in Definition 2.1.5. The maximal class condition, necessary for the construction of the canonical frame in the symplectification procedure, is that this flag completely saturates the space in which it belongs. Our work is dedicated to developing the tools necessary to prove that this condition holds, or can be obtained via some reduction process.

It turns out that we can more succinctly describe the maximal class condition in purely control-theoretic language. Our first main result is that the maximal class condition is equivalent to a property of the critical points of the endpoint map; in particular, a distribution is of maximal class if and only if the endpoint map along an abnormal extremal has corank-1. This fact is already well-known for rank-2 bracket-generating distributions and we directly extend the proof in [30] to one for distributions of arbitrary rank.

Our second main result makes use of properties of the Tanaka symbol. Every symbol of a distribution at a point is equipped with the *flat*, left-invariant distribution on its simply-connected Lie group, obtained by translations of the first graded component of the symbol. In Section 2.3 we prove that to determine if a distribution is of maximal class at a point it suffices to prove that the flat distribution of its symbol at that point is of maximal class. Then, in Section 2.4 we prove our third main result, that the abnormal extremals of a flat distribution of a symbol satisfying generic transversality conditions project onto the abnormal extremals of the flat distributions of its quotient symbols. These results suggest the following strategy for proving the maximal class conjecture:

1. prove that the flat distribution of the free symbol with r generators and degree of nilpotency μ is of maximal class;
2. prove that the flat distributions of all symbols obtained as quotients of the free symbol are of maximal class;
3. conclude that all distributions with symbol at a point isomorphic to the free symbol, or a quotient of it, are of maximal class.

We present our fourth main result in Sections 3.2 and 3.3, where we complete the first step in this strategy for $r = 2$ and $\mu = 4$ and 5. However, it remains to show that the maximal class condition is preserved under projection onto a quotient symbol.

We provide an example of an alternative strategy to proving a distribution is of maximal class at the end of Chapter 3. In Section 3.4 we consider rank-2 distributions canonically associated with Monge ordinary differential equations of form (3.12) and, building on work by Ian Anderson and Boris Kruglikov [9], prove that these Monge distributions are all of maximal class. We thus obtain one of the main results of [9] as an immediate consequence of an application of the symplectification procedure in Theorem 1.1 of Doubrov and Zelenko in [18].

We collect our results on arbitrary rank distributions in Chapter 2 and restrict our attention to rank-2 distributions in Chapter 3. It was shown in [30] that all distributions of rank 2 and with dimension $n \leq 6$ are of maximal class. No examples of distributions with non-maximal class have yet been found, and our primary focus in this thesis is on developing tools for proving the maximality of class of distributions of type $(2, n)$ with small growth vector $(2, 3, 5, \ldots)$ and $n \geq 6$.

2. THE CLASS OF A RANK-r DISTRIBUTION

Let D be a smooth distribution of rank r on a n-dimensional manifold M. The *filtration* of D at $q \in M$ is a sequence of nested subspaces of T_qM generated by the iterated Lie brackets of vector fields in D:

$$D(q) = D^1(q) \subset D^2(q) \subset \cdots \subset D^{\mu(q)}(q) \subseteq T_qM, \tag{2.1}$$

where $D^i(q) = D^{i-1}(q) + [D, D^{i-1}](q)$ for $i > 1$, and the function $\mu(q)$ indicates where the filtration stabilizes. The corresponding dual object is a filtration in the cotangent fiber T_q^*M with each subspace consisting of covectors annihilating (2.1):

$$(D^k)^{\perp} = \{\lambda = (p, q) \in T^*M \mid \langle p, v\rangle = 0 \text{ for all } v \in D^k(q)\}, \tag{2.2}$$

with $(D^{\mu(q)})^{\perp}$ equal to the zero-section of TM. We say that D is *bracket-generating* if there exists an integer $\mu(q) \geq 2$ such that $D^{\mu(q)}(q) = T_qM$, in which case we call $\mu(q)$ the *degree of nonholonomy* of D at q. The *small growth vector* at $q \in M$ of a distribution D is a non-decreasing sequence of positive integers $r(q) = (r_1(q), \ldots, r_{\mu(q)}(q))$, such that $r_i(q) = \dim D^i(q)$ is the dimension of the i^{th} subspace of (2.1). We say that D is *regular* at q if the growth vector is constant on a neighborhood of q. By restricting our attention to a sufficiently small neighborhood of a regular point, if necessary, we henceforth work exclusively with regular, bracket-generating distributions of rank r on an n-dimensional manifold and, for brevity, simply refer to them as (r, n)-distributions.

The class of a regular, bracket-generating distribution D at a point $q \in M$ is determined by the existence of a nonempty subset of the fiber $T_\lambda T^*M$, where $\lambda = (p, q) \in T_q^*M$ is a point in a special curve in T^*M intrinsically determined by the data defining the distribution. These special curves are the so-called *characteristic curves* or *abnormal trajectories* of a distribution. We define them and the subset of interest, and then finally provide a definition of the class of a distribution in Definition 2.1.7.

2.1 The class of a regular, bracket-generating distribution

Let $\pi : T^*M \to M$ denote the canonical projection and $\pi_* : TT^*M \to TM$ its differential. Denote the *tautological Liouville 1-form* on M by $\mathfrak{s} \in \Lambda^1(T^*M)$ such that for all $\lambda = (p, q) \in T_q^*M$ and $v \in T_\lambda T^*M$, $\mathfrak{s}(\lambda)(v) = (\lambda \circ \pi_*)(v) = \langle p, \pi_*(v)\rangle_q$. The *canonical symplectic 2-form* is the exterior derivative of the tautological 1-form, $\sigma :=\, \mathrm{d}\mathfrak{s} \in \Lambda^2(T^*M)$.

A *horizontal curve* of D is an absolutely continuous curve $x : [0, T] \to M$ that is everywhere tangent to D, *i.e.*, $\dot{x}(t) \in D(x(t))$ for all $t \in [0, T]$. We are interested in horizontal curves satisfying the following properties.

Definition 2.1.1: An *abnormal trajectory* or *characteristic curve of D* is a horizontal curve $x : [0, T] \to M$ that is the projection under π of an absolutely continuous curve $\gamma : [0, T] \to T^*M$ satisfying $\gamma(t) = (\psi(t), x(t))$ and

(A1) $\gamma \in D^\perp$ for all $t \in [0, T]$, and

(A2) $\dot{\gamma} \in \mathrm{Ker}(\sigma|_{D^\perp})$ a.e. in $[0, T]$.

The curve γ is called an *abnormal extremal* or an *abnormal lift of x*.

We can describe $\mathrm{Ker}\,\sigma|_{D^\perp}$ using Hamiltonian vector fields on $T_\lambda T^*M$. Given a local section $X \in \Gamma(TM)$ define its *quasi-impulse* to be the function $u(\lambda) := \langle p, X(q)\rangle$, $\lambda = (p, q)$. The respective quasi-impulses u_X, u_Y and $u_{[X,Y]}$ of local sections $X, Y, [X, Y] \in \Gamma(TM)$, satisfy

$$\vec{u}_X(u_Y) =\, \mathrm{d}u_Y(\vec{u}_X) = u_{[X,Y]}, \tag{2.3}$$

$$[\vec{u}_X, \vec{u}_Y] = \vec{u}_{[X,Y]}. \tag{2.4}$$

We can construct Hamiltonian vector fields from quasi-impulses. Given a vector field $X \in \Gamma(TM)$ with quasi-impulse u_X, define its *Hamiltonian lift* to be the unique vector field $\vec{u}_X \in \Gamma(TT^*M)$ satisfying

$$\sigma(\cdot, \vec{u}_X) = \mathrm{d}u_X(\cdot) \tag{2.5}$$

We will use the following local coordinates for the remainder of this section. Let $\{X_j\}_{j=1}^r$ be a local basis for D at $q \in M$ and complete it to a basis $\mathcal{X} := \{X_i\}_{i=1}^n$ for T_qM. Let u_k denote the quasi-impulse associated with basis element $X_k \in \mathcal{X}$. The set of all quasi-impulses $\{u_i\}_{i=1}^n$ associated to vector fields in $\mathcal{X}$ forms local coordinates on T^*M, and thus we have a local basis $\{\partial u_i\}_{i=1}^n$ for $T_\lambda(T^*_{\pi(\lambda)}M)$. In these coordinates, the Hamiltonian lift of u_i is given by

$$\begin{aligned}\vec{u}_i &= X_i + \sum_{j=1}^n \vec{u}_i(u_j)\,\partial u_j \\ &= X_i + \sum_{j=1}^n \sum_{k=1}^n c_{ji}^k u_k\,\partial u_j,\end{aligned} \tag{2.6}$$

where the functions $\{c_{ji}^k\}_{i,j,k=1}^n$ are the structure functions associated with $\mathcal{X}$. In these local coordinates we have

$$D^\perp = \{\lambda \in T^*M \mid u_1(\lambda) = \cdots = u_r(\lambda) = 0\}, \tag{2.7}$$

$$T_\lambda D^\perp = \{v \in T_\lambda T^*M \mid \mathrm{d}u_1(v) = \cdots = \mathrm{d}u_r(v) = 0\}, \tag{2.8}$$

and define

$$H_D(\lambda) := \mathrm{span}\{\vec{u}_i\}_{i=1}^r(\lambda). \tag{2.9}$$

It is straightforward to show that $H_D(\lambda)$ is well-defined independently of this choice of coordinates. The following is Lemma 2.1 in [29].

Lemma 2.1.2 (Zelenko (1999)): If D is a (r, n)-distribution then

$$\mathrm{Ker}(\sigma|_{D^\perp})(\lambda) = T_\lambda D^\perp \cap H_D(\lambda) \tag{2.10}$$

We are interested in the case when the subspaces in (2.10) intersect transversally at $\lambda \in D^{\perp}$ in a 1-dimensional subspace.

Definition 2.1.3: The abnormal extremal γ is called a *regular abnormal extremal* if for all $\lambda \in \gamma$,

(A3) $\dim \mathrm{Ker}(\sigma|_{D^{\perp}}(\lambda)) = 1$.

The set of all regular abnormal extremals of D is denoted W_D, with $W_D(q) := W_D \cap T_q^*M$. The set of points at which $\sigma|_{D^{\perp}}$ is degenerate is the *characteristic line bundle of* D, denoted $\mathcal{C}_D := \mathrm{Ker}\,\sigma|_{D^{\perp}}$, and a vector field $\vec{h} \in \mathcal{C}_D$ is called a *characteristic vector field* of D.

In Chapter 3 we describe the set W_D for all $(2, n)$-distributions D. See [32] for full details for distributions of arbitrary rank. We henceforth restrict our attention to those (r, n)-distributions D for which W_D is a nonempty submanifold of $D^{\perp}$. This space is foliated by the 1-foliations of $\mathcal{C}_D$, *i.e.* the regular abnormal extremals.

We now define the pullback of the distribution D to a distribution on TT^*M via the canonical cotangent bundle projection. As we will see, the pullback encodes information about the geometry of the distribution itself and possesses additional useful geometric structure. See [32] for more information and discussion. This construction is central to our definition of the class of a distribution.

Definition 2.1.4: The *lifted distribution* or *lift of* D is the pullback of D over the set of regular abnormal extremals:

$$\mathcal{J}(\lambda) := (\pi^* D)(\lambda) = \{v \in T_\lambda W_D \mid \pi_* v \in D(\pi(\lambda))\}. \tag{2.11}$$

We study the lifted distribution because it contains distinguished subdistributions and permits recovery of the distribution itself. For all $\lambda \in T_\lambda W_D$ define the distribution

$$V(\lambda) := \{v \in T_\lambda W_D \mid \pi_*(v) = 0\} \tag{2.12}$$

and observe that for all $\lambda \in W_D(q)$,

$$V(\lambda) + \mathcal{C}_D(\lambda) \subset \mathcal{J}(\lambda). \tag{2.13}$$

Our definition of the class of a distribution will depend on a flag of subspaces in $T_\lambda W_D$ starting from the distribution $\mathcal{J}(\lambda)$.

Definition 2.1.5: The *flag of the lifted distribution* is the flag of subspaces

$$\mathcal{J}(\lambda) =: \mathcal{J}^{(0)}(\lambda) \subset \mathcal{J}^{(1)}(\lambda) \subset \mathcal{J}^{(2)}(\lambda) \subseteq \mathcal{J}^{(3)}(\lambda) \subseteq \cdots \subset T_\lambda T^*M \tag{2.14}$$

recursively defined for all $i \geq 0$

$$\mathcal{J}^{(i+1)}(\lambda) = \mathcal{J}^{(i)}(\lambda) + [\mathcal{C}_D, \mathcal{J}^{(i)}](\lambda), \tag{2.15}$$

where $[\mathcal{C}_D, \mathcal{J}^{(i)}] = \left\{[\vec{h}, \vec{v}] \mid \vec{h} \in \mathcal{C}_D,\ \vec{v} \in \mathcal{J}^{(i)}\right\}$.

The following lemma establishes an upper bound on the dimension of the flag of the lifted distribution. First, note that the kernel of the tautological Liouville 1-form defines a corank-1 distribution

$$\Delta(\lambda) := \{v \in T_\lambda T^*M \mid \mathfrak{s}(\lambda)(v) = 0\}, \tag{2.16}$$

and recall, c.f. [20, 32], that the *Cauchy characteristic subdistribution* of a distribution S is the maximal sub-distribution H in S satisfying $[H, S] \subset S$.

Lemma 2.1.6: $\mathcal{C}_D = \operatorname{Ker} \sigma|_{D^\perp}$ *is the Cauchy characteristic of* Δ *and* $\mathcal{J}^{(i)}(\lambda) \subset \Delta(\lambda)$ *for all* $i \geq 0$*.*

Proof. Note that for local sections $X, Y \in \Gamma(\Delta)$, Cartan's formula [15] implies that $\sigma(\lambda)(X, Y) = -\mathfrak{s}(\lambda)([X, Y])$, which proves the first statement. For the second, note that $\mathcal{J}^{(0)} \subset \Delta$ by construction. Since $\mathcal{J}^{(1)} = \mathcal{J}^{(0)} + [\mathcal{C}_D, \mathcal{J}^{(0)}]$ and $\mathcal{C}_D \subset \Delta$, a straightforward induction implies $\mathcal{J}^{(i)} \subset \Delta$ for all i. □

Define the integer-valued function

$$\nu(\lambda) := \min\left\{0 \le i < 2n \mid \mathcal{J}^{(i+1)}(\lambda) = \mathcal{J}^{(i)}(\lambda)\right\}, \tag{2.17}$$

describing where the flag of $\mathcal{J}$ first stabilizes at each λ. By the Lemma above, the maximum value of ν is determined by the largest $i < 2n$ such that $\mathcal{J}^{(i)}(\lambda) = \Delta(\lambda)$. We will show in Chapter 3 that $3 \le \nu(\lambda) \le n-3$ for any $(2,n)$-distribution D and for all $\lambda \in W_D$.

We can finally define the class of a distribution at points of M.

Definition 2.1.7: The *class of a distribution* D at $q \in M$ is the function

$$m(q) = \max\left\{\nu(\lambda) \mid \lambda \in W_D(q)\right\}.$$

Let $\mathcal{R}_D \subset W_D$ denote the subset of points λ at which the flag of $\mathcal{J}^{(0)}(\lambda)$ saturates its tangent subspace:

$$\mathcal{R}_D = \{\lambda \in W_D \mid \mathcal{J}^{(i)}(\lambda) = \Delta(\lambda) \text{ for some } i < 2n\}$$

We say that D is of *maximal class* at q if and only if the subset $\mathcal{R}_D(q) := \mathcal{R}_D \cap T_q^*M$ is nonempty.

It was shown in Proposition 3.4 of [30] that the subset $\mathcal{R}_D(q)$ of a $(2,n)$-distribution D is the complement of the zero set of a polynomial in the quasi-impulses $u_{r+1}, \ldots, u_n$. Let $\mathbb{C}[u_{r+1}, \ldots, u_n]$ denote the ring of polynomials in the quasi-impulses associated with a local basis for some (r,n)-distribution D. The proof of the following Lemma is the straightforward extension of the proof of Proposition 3.4 in [30] to distributions of arbitrary rank.

Lemma 2.1.8: If D is a (r,n)-distribution then $\mathcal{R}_D(q)$ is the complement in $W_D(q)$ of an algebraic variety defined by a polynomial in $\mathbb{C}[u_{r+1}, \ldots, u_n]$.

Proof. Since $T_\lambda W_D(q) \subset T_\lambda D^\perp(q)$, it follows from equation (2.8), equation (2.13) and Lemma 2.1.2 that a basis for $\mathcal{J}^{(0)}(\lambda) \subset T_\lambda W_D(q)$ consists of linear combinations of vector fields ∂u_j and $\vec{u}_i$, $j \in \{r+1, \ldots, n\}$ and $i \in [n]$. Likewise, any characteristic vector field $\vec{h} \in \mathcal{C}_D$ can be written as a linear combination of Hamiltonian lifts $\vec{u}_1, \ldots, \vec{u}_r$ with coefficients in $\mathbb{C}[u_{r+1}, \ldots, u_n]$, from equations (2.8) and (2.9). Thus, any local basis for a subspace $\mathcal{J}^{(k+1)} = \mathcal{J}^{(k)} + [\vec{h}, \mathcal{J}^{(k)}]$ consists of linear combinations of the ∂u_j and $\vec{u}_i$ with coefficients that are polynomial in both the quasi-impulses and structure functions associated to $\mathcal{X}$ in $\mathbb{C}[u_{r+1}, \ldots, u_n]$. But since the structure functions are constant on $D^\perp(q) \subset T^*_q M$, the condition $\Delta(\lambda) = \mathcal{J}^{(k)}(\lambda)$ for some $k < 2n$ is determined by a polynomial in the quasi-impulses. In particular, this polynomial is a minor of a matrix with entries in $\mathbb{C}[u_{r+1}, \ldots, u_n]$. $\square$

We require this Lemma in the proof of Proposition 2.3.2, where we show it suffices to reduce our analysis of the maximal class condition to the case of so-called "flat" distributions. In the next section we provide some partial motivation for our study of the class of a distribution.

2.2 The corank of an abnormal trajectory

If $\gamma = (\psi, x) \in D^\perp$ is an abnormal lift of x then it follows from (A1) that $c\gamma := (c\psi, x) \in D^\perp$ for some number $c \neq 0$. It is shown in [4, 6] that the space of lifts of a given abnormal trajectory is a vector space of dimension at least 1, due essentially to condition (A1) and Pontryagin's Maximum Principle.

Our objective in this section is to show that the points of maximal class of D coincide exactly with points of certain abnormal trajectories of D whose space of lifts is exactly 1-dimensional. To this end we to introduce a special input-to-state mapping called the *endpoint map*.

First, recall that in the previous section we defined a horizontal curve x of a distribution D to be curve on M everywhere tangent to the distribution. If $D(q) = \operatorname{span}\{X_i\}_{i=1}^r$ is a local basis for D at q, then locally near q we can write a horizontal curve x as the solution to the differential equation $\dot{x}(t) = \sum_{i=1}^r X_i(x(t))u_i(t)$, where $u = (u_1, \ldots, u_r)$ is a vector of scalar, time-varying functions called *controls* or *inputs*.

Definition 2.2.1: The *endpoint map* is the function $F : [0,T] \times L_\infty^r[0,T] \to M$ sending pairs (t, u), where $u \in L_\infty^r[0,T]$ is a time-varying control function and t is a *final time*, to the point in the horizontal curve $x : [0,T] \to M$ at time t, defined $F(t, u) := x(t)$.

This leads us to a characterization of abnormal trajectories that is well-known to be equivalent to Definition 2.1.1.

Proposition 2.2.2: A horizontal curve x with control $u \in L_\infty^r[0,T]$ is an abnormal trajectory if and only if $k = \operatorname{codim}\operatorname{Im}(DF(T,u)) \geq 1$, in which case x is called a corank-k abnormal trajectory. Furthermore, if (T,u) is a critical point of the endpoint map then so is (t,u) for all $t \in [0,T]$.

Thus, a horizontal curve $x : [0,T] \to M$ with control u is an abnormal trajectory if and only if the pair (T,u) is a critical point of the endpoint map. Further discussion and proofs of the equivalence of this definition with Definition 2.1.1 can be found in [22, 4, 6].

Let $DF(T,u)$ denote the derivative of the endpoint map. We will use the notation and formalisms of the *chronological calculus* in order to derive an expression for the derivative at critical points. The chronological calculus is an operator calculus that enables the identification of points, diffeomorphisms and vector fields on a manifold with operators in the algebra $C^\infty(M)$ of smooth functionals on M. We take advantage of Volterra series expansions and other formula naturally expressed in the chronological calculus in our derivation of the endpoint map of an abnormal trajectory and its derivative in Proposition 2.2.3. According to [3], a chief advantage of this operator calculus is that it enables one to reduce analysis on a nonlinear space, the smooth manifold M, to analysis in an (infinite-dimensional) linear space, the algebra $C^\infty(M)$.

For the remainder of this section we work in local coordinates defined as follows: at any point q in an abnormal trajectory $\hat{x}$ take a vector field X_1 tangent to the abnormal trajectory. Completing this vector field to a local basis $\{X_1, \ldots, X_r\}$ for D on a neighborhood U of q, and on a sufficiently small interval $[0, T]$ of the abnormal trajectory in U write $\dot{\hat{x}}(t) = X_1(\hat{x}(t))$ for all $t \in [0, T]$. By construction, in these local coordinates the pair $(t, \hat{u})$ is a critical point of the endpoint map for all $t \in [0, T]$, where $\hat{u} = (1, 0, \ldots, 0) \in L^r_\infty[0, T]$.

The following Proposition allows us to describe the critical points of the endpoint map in these special coordinates aligned with the flow of an abnormal extremal. In particular, using the chronological calculus and following the approach in Section 4 of [4] and also [5, 6], we derive the so-called *first variation* of the endpoint map, which is the derivative $DF_t : L^r_\infty[0, T] \to T_{F_t(q)}M$ of the endpoint map $F_t(\cdot) := F(t, \cdot)$ with time given and fixed.

Proposition 2.2.3: *Let $\hat{x} : [0, T] \to M$ be an abnormal trajectory in the given coordinates, with control $\hat{u} = (1, 0, \ldots, 0)$, initial condition $\hat{x}(0) =: \hat{x}_0$ and endpoint $\hat{x}(t) =: \hat{x}_1$. Then, the first variation of $F_t := F(t, \cdot)$, for any $t \in [0, T]$, at the critical point $(t, \hat{u})$ is*

$$DF_t(\hat{u}) \cdot u \;=\; \sum_{i=1}^{r} X_i(\hat{x}_1)\, v_i(t) \;-\; \int_0^t \sum_{i=2}^{r} \sum_{j=1}^{n} \frac{(\tau - t)^{j-1}}{(j-1)!}\, (\mathrm{ad}\, X_1)^j (X_i)\, v_i(\tau)\, d\tau, \quad (2.18)$$

where $u = \hat{u} + \dot{v}$, and $\dot{v}(\tau) = (\dot{v}_1(\tau), \ldots, \dot{v}_r(\tau)) \in L^r_\infty[0, T]$ are arbitrary controls.

Proof. The abnormal extremal is written in the chronological calculus as $\hat{x}(t) = \hat{x}_0 \circ e^{t\,X_1}$. Note that $u = \hat{u} + \dot{v} = (1 + \dot{v}_1, \dot{v}_2, \ldots, \dot{v}_r)$. The horizontal trajectory satisfies differential equation

$$\dot{x} = f(x,u) = X_1(x)(1 + \dot{v}_1) + \sum_{i=2}^{r} X_i(x)\dot{v}_i = X_1(x) + \sum_{i=1}^{r} X_i(x)\dot{v}_i$$

and consequently

$$\begin{aligned} F(t,u) &= \hat{x}(0) \circ \overrightarrow{\exp} \int_0^t (X_1 + \sum_{i=1}^{r} X_i\, \dot{v}_i(\tau))\, \mathrm{d}\tau \\ &= \hat{x}(0) \circ e^{t\,X_1} \circ \overrightarrow{\exp} \int_0^t \mathrm{Ad}\, e^{(\tau - t)\,X_1} \cdot (\sum_{i=1}^{r} X_i\, \dot{v}_i(\tau))\, \mathrm{d}\tau \\ &= \hat{x}(t) \circ \overrightarrow{\exp} \int_0^t e^{(\tau - t)\ \mathrm{ad}\,X_1} \cdot (\sum_{i=1}^{r} X_i\, \dot{v}_i(\tau))\, \mathrm{d}\tau, \end{aligned}$$

where in the second equality we used the left generalized variational formula (A.3) and in the third we used (A.4). Since $e^{t\,\mathrm{ad}\,X} \cdot X = X$ for any vector field X,

$$\begin{aligned} &= \hat{x}(t) \circ \overrightarrow{\exp} \int_0^t \Big(X_1\, \dot{v}_1(\tau) + e^{(\tau - t)\ \mathrm{ad}\,X_1} \cdot (\sum_{i=2}^{r} X_i\, \dot{v}_i(\tau)) \Big)\, \mathrm{d}\tau, \\ &= \hat{x}(t) \circ \overrightarrow{\exp} \int_0^t \Big(X_1\, \dot{v}_1(\tau) + Y_{t,\tau} u(\tau) \Big)\, \mathrm{d}\tau, \end{aligned} \tag{2.19}$$

where we defined

$$Y_{t,\tau} u(\tau) := e^{(\tau - t)\,\mathrm{ad}\,X_1} \cdot (\sum_{i=2}^{r} X_i \dot{v}_i(\tau)).$$

Observe that

$$\begin{aligned} \frac{dY_{t,\tau}}{dt} u(\tau) &= (-\,\mathrm{ad}\,X_1)\Big(e^{(\tau - t)\ \mathrm{ad}\,X_1} \cdot (\sum_{i=2}^{r} X_i\, \dot{v}_i(\tau)) \Big) \\ &= -\Big(\mathrm{ad}\,X_1 + (\tau - t)(\mathrm{ad}\,X_1)^2 + \frac{(\tau - t)^2}{2}(\mathrm{ad}\,X_1)^3 + \cdots \Big)\Big(\sum_{i=2}^{r} X_i \dot{v}_i(\tau) \Big), \\ &= -\left(\sum_{j=1}^{n} \sum_{i=2}^{r} \frac{(\tau - t)^{j-1}}{(j-1)!} (\mathrm{ad}\,X_1)^j (X_i) \dot{v}_i(\tau) \right) \end{aligned} \tag{2.20}$$

where we applied the asymptotic Volterra series expansion (A.1) to the chronological exponent in the second equality. Using the Volterra expansion on the outermost chronological exponent in (2.19), applying integration by parts, and evaluating at $(t, \hat{u})$ yields

$$
\begin{aligned}
DF(t, \hat{u}) \cdot (\delta t, u) &= X_1(\hat{x}_1)\, v_1(t) + \int_0^t Y_{t,\tau}\, u(\tau)\, \mathrm{d}\tau, \\
&= \sum_{i=1}^{r} \Big(X_i(\hat{x}_1)\, v_i(t) \Big) + \int_0^t \frac{dY_{t,\tau}}{dt} v(\tau)\, \mathrm{d}\tau.
\end{aligned}
$$

Substitution of (2.20) yields the desired result. □

We require the following result regarding the projection of the flag of the lifted distribution from $T_\lambda T^*M$ to $T^*_{\pi(\lambda)}M$ in these local coordinates. Our proof is a straightforward extension of the proof of Proposition 3.3 in [30] to distributions of arbitrary rank. Recall the definition of the subspace $V(\lambda)$ in equation (2.12).

Lemma 2.2.4: *For all* $\lambda \in W_D$,

$$
\pi_*(\mathcal{J}^{(i)}(\lambda)) = \operatorname{span}\Big\{ (\operatorname{ad} X_1)^k (X_j) \mid j \in [r],\, 0 \le k \le i \Big\}
$$

and $\dim \mathcal{J}^{(i)}(\lambda) = \dim V(\lambda) + \dim \pi_*(\mathcal{J}^{(i)}(\lambda))$.

Proof. Let $\widetilde{X}_1$ be a local section of $\mathcal{C}_D$ tangent to a sufficiently small segment of regular abnormal extremal lift $\gamma : [0, T] \to W_D \subset D^\perp$ of the abnormal trajectory $\hat{x} = \pi(\gamma)$, and construct local sections $\{\widetilde{X}_i\}_{i=2}^r$ satisfying $\pi_*(\operatorname{span}\{\widetilde{X}_i\}_{i=1}^r) = D(\pi(\lambda))$. Observe that $V(\lambda) = T_\lambda(T^*_{\pi(\lambda)}M) \cap T_\lambda W_D$. The lift of the distribution is $\mathcal{J}^{(0)}(\lambda) = V(\lambda) + \operatorname{span}\{\widetilde{X}_i\}_{i=1}^r$ and thus

$$
\mathcal{J}^{(i)}(\lambda) = V(\lambda) + \operatorname{span}\{(\operatorname{ad} \widetilde{X}_1)^k (\widetilde{X}_j) \mid j \in [r], 0 \le k \le i\}(\lambda). \tag{2.21}
$$

Take an n-dimensional submanifold $E \subset D^\perp$ containing γ and intersecting the fibers $W_D(q)$ transversally at every point $\lambda \in \gamma$. There exists a sufficiently small neighborhood $\widetilde{E} \subset E$ of γ which π maps bijectively to some neighborhood $V \subset M$ of the abnormal trajectory x. It follows that for all $\lambda \in \widetilde{E}$ that $\pi_*(\widetilde{X}_i(\lambda)) = X_i(\pi(\lambda))$, yielding the desired result.

Suppose that $T_\lambda W_D$ has codimension $m \geq 0$ in $T_\lambda D^\perp$, so that $\dim T_\lambda W_D = 2n - (m+r)$ and $\dim \Delta(\lambda) = 2n-(m+r+1)$. Then, from the above, $\dim \mathcal{J}^{(0)}(\lambda) = n-m$ and thus $\dim \mathcal{J}^{(i)}(\lambda) = \dim V(\lambda) + \dim \pi_*(\mathcal{J}^{(i)}(\lambda))$. $\square$

We can finally state our first result, that a distribution is of maximal class at a point if and only if it possesses a corank-1 abnormal trajectory passing through that point. Our proof is a straightforward extension of the ideas outlined in Remark 3.2 in [30].

Theorem 2.2.5: A regular, bracket-generating (r,n)-distribution D on a manifold M is of maximal class at $q \in M$ if and only if there exists a corank-1 abnormal trajectory of D through $q \in M$ that lifts to a regular abnormal extremal.

Proof. Let $\hat{x} : [0,T] \to M$ be a corank-k abnormal trajectory with control $\hat{u}$, such that $(t, \hat{u})$ is a critical point at which $DF_t(\hat{u})$ has rank $n-k$. Then there exists a covector $\psi \in T^*_{\hat{x}(t)}M$ such that $\langle \psi, DF_t(\hat{u}) \cdot u \rangle = 0$ for all $u \in L^r_\infty[0,T]$. However, from Proposition 2.2.3 and the Dubois-Raymond Lemma (see Section 4 of [6] for details), and Lemma 2.2.4, ψ annihilates the subspace

$$\pi_*(\mathcal{J}^{(n)}(\lambda)) = \operatorname{span}\{X_i\}_{i=1}^r + \operatorname{span}\{(\operatorname{ad} X_1)^j(X_i) \mid j \in [n],\ 2 \leq i \leq r\}(\pi(\lambda)),$$

where λ is a point in the regular abnormal extremal lift $\gamma : [0,T] \to W_D$ of $\hat{x}$. It follows that $\dim(\pi_*(\mathcal{J}^{(n)}(\lambda))) = n-k$ for all λ such that $\pi(\lambda) = \hat{x}(t)$. This proves $\dim \pi_*(\mathcal{J}^{(n)}(\lambda)) = n-k$ if and only if $\hat{x}(t) = \pi(\lambda)$ is a point in a corank-k abnormal trajectory.

It follows from Lemma 2.2.4 that if $\dim \pi_*(\mathcal{J}^{(i)}(\lambda)) = n-1$ then $\dim \mathcal{J}^{(i)}(\lambda) = n-(m+r)+n-1 = 2n-(m+r+1)$ if and only if the distribution is of maximal class at $\hat{x}(t)$. $\square$

2.3 Reduction to the case of flat distributions

The symbol of a distribution is the graded object associated to the filtration (2.1) of an (r,n)-distribution D at a point $q \in M$. We define this object below and refer to [31] for further details and discussion. Our objective in this section is to prove that a distribution D is of maximal class at $q \in M$ if the flat, left-invariant distribution $D_{\mathfrak{g}}$ of its symbol $\mathfrak{g}$ at q is of maximal class. We define these objects below.

Proposition/Definition 2.3.1: The *symbol* of D at $q \in M$ is the graded vector space associated to filtration (2.1) at the point $q \in M$, such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{\mu(q)} \tag{2.22}$$

where $\mathfrak{g}_1 = D(q)$ and $\mathfrak{g}_{i+1} = D^{i+1}(q)/D^i(q)$ for all $i \in [\mu - 1]$, with canonical projections $\mathfrak{p}_{i+1} : D^{i+1} \to \mathfrak{g}_{i+1}$ onto each subspace of the symbol and the structure of a graded, nilpotent Lie algebra defined as follows: let $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ and take local sections $\widetilde{X} \in \Gamma(D^i)$ and $\widetilde{Y} \in \Gamma(D^j)$ such that $\mathfrak{p}_i(\widetilde{X}) = X$ and $\mathfrak{p}_j(\widetilde{Y}) = Y$. Then, the Lie bracket on $\mathfrak{g}$ is defined

$$[X, Y] = \mathfrak{p}_{i+j}([\widetilde{X}, \widetilde{Y}]).$$

We say a symbol of a distribution D on M is *constant* if it is isomorphic to the same graded nilpotent Lie algebra at all points of M.

Let $M(\mathfrak{g})$ denote the simply-connected Lie group of the symbol $\mathfrak{g}$ and let $e \in M(\mathfrak{g})$ be the identity element. The first graded component of $\mathfrak{g}$ defines a distribution $D_{\mathfrak{g}}$ called the *flat, left-invariant distribution of* $\mathfrak{g}$, such that $D_{\mathfrak{g}}(e) := \mathfrak{g}_1$.

Let D have small growth vector $(r_1, \ldots, r_\mu)$. Note that if $\{X_i\}_{i=1}^r$ is a local basis for the distribution at $q \in M$ then it can always be completed to a local basis $\{X_i\}_{i=1}^n$ for T_qM, with the property that $\operatorname{span}\{X_j\}_{j=1}^{r_i} = D^i$. Such local bases are said to be *compatible with the filtration* (2.1) of D. Let $\mathcal{X}$ denote such a compatible local basis at $q \in M$ and $\{c_{ji}^k\}_{i,j,k=1}^n$ its associated structure functions. Define the *weight* of a basis element $X_j \in \mathcal{X}$ to be the integer $i \in [\mu]$ such that $r_{i-1} \leq j \leq r_i$ and write $\mathrm{wt}(X_j) = i$.

Construct a basis $\overline{\mathcal{X}} := \{\overline{X}_i\}_{i=1}^n$ for $\mathfrak{g}$ by projecting each basis element of the compatible basis $\mathcal{X}$ by its appropriate canonical projection: if $X_i \in D^j$ then $\overline{X}_i = \mathfrak{p}_j(X_i) = X_i \bmod D^{j-1}$. We say that a basis $\overline{\mathcal{X}}$ is *compatible with the grading* on $\mathfrak{g}$ if $\mathrm{span}\{X_j\}_{j=r_{i-1}}^{r_i} = \mathfrak{g}_i$. The *weight* of $\overline{X}_j \in \overline{\mathcal{X}}$ is defined to be the integer $i \in [\mu]$ such that $X_j \in \mathfrak{g}_i$ and written $\mathrm{wt}(\overline{X}_j) = i$. It follows that if $\{\overline{c}_{ji}^k\}$ are the structure constants associated with the compatible basis $\overline{\mathcal{X}}$ then $\overline{c}_{ji}^k \equiv 0$ if $\mathrm{wt}(\overline{X}_j) + \mathrm{wt}(\overline{X}_i) \neq \mathrm{wt}(\overline{X}_k)$ and otherwise $\overline{c}_{ji}^k = c_{ji}^k(q)$.

The following result forms the basis of our claim that it suffices to consider if the flat distribution of the symbol of the distribution at a point is of maximal class, in order to determine if the distribution itself is of maximal class at that point.

Proposition 2.3.2: Let $\mathfrak{m}$ be a constant symbol with flat, left-invariant distribution $D_{\mathfrak{m}}$ and D be an (r,n)-distribution with symbol $\mathfrak{g}$ at $q \in M$ isomorphic to $\mathfrak{m}$ as graded nilpotent Lie algebras. If $D_{\mathfrak{m}}$ is of maximal class then D is of maximal class at $q \in M$.

Proof. First, note that if $D_{\mathfrak{m}}$ is of maximal class at the identity element $e \in M(\mathfrak{m})$ then it is of maximal class everywhere on $M(\mathfrak{m})$. Let $W_{\mathfrak{m}} \subset D_{\mathfrak{m}}^{\perp}$ and $W_D \subset D^{\perp}$ denote the subsets foliated by the regular abnormal extremals of $D_{\mathfrak{m}}$ and D, respectively. Define also $\mathcal{R}_D(e)$ and $\mathcal{R}_D(q)$ to be the subsets at which the flags of their lifted distributions saturate at the corank-1 distribution Δ defined by equation (2.16).

By Lemma 2.1.8, there exist polynomials $a_{\mathfrak{m}}$ and a_D such that $\mathcal{R}_{\mathfrak{m}}(e)$ and $\mathcal{R}_D(q)$ are the complements of the zero sets of each polynomial, respectively. We want to show that $a_{\mathfrak{m}} \not\equiv 0$ implies $a_D \not\equiv 0$. Let $\{u_i\}_{i=1}^n$ and $\{\overline{u}_i\}_{i=1}^n$ denote the quasi-impulses associated with compatible bases $\mathcal{X}$ and $\overline{\mathcal{X}}$ for D and $D_{\mathfrak{m}}$, respectively, where $\overline{\mathcal{X}}$ is obtained by projecting each basis element of $\mathcal{X}$ to the symbol. The polynomial a_D is the sum of polynomials of the form $\alpha_b u^b$, where α_b is a polynomial in the structure functions $\{c_{ji}^k\}_{i,j,k=1}^n$ associated with $\mathcal{X}$, $b = (b_{r+1}, \ldots, b_n)$ is an exponent vector, and $u^b = u_{r+1}^{b_{r+1}} \cdots u_n^{b_n}$ is a monomial in the quasi-impulses.

Define a ring homomorphism φ that maps polynomials in the quasi-impulses and structure functions associated with $\mathcal{X}$ to those associated with $\overline{\mathcal{X}}$, such that $\varphi(u_i) = \overline{u}_i$ and $\varphi(c_{ji}^k) = \overline{c}_{ji}^k$, where $\overline{c}_{ji}^k = 0$ if and only if $\mathrm{wt}(X_j) + \mathrm{wt}(X_i) \neq \mathrm{wt}(X_k)$. It follows that $\varphi(a_D) = a_{\mathfrak{m}}$. Since $D_{\mathfrak{m}}$ is of maximal class if and only if $a_{\mathfrak{m}} \not\equiv 0$, there exists at least one polynomial $\alpha_b u^b$ in a_D such that α_b is a nonzero polynomial in structure functions c_{ji}^k, where i, j and k satisfy $\mathrm{wt}(X_j) + \mathrm{wt}(X_i) = \mathrm{wt}(X_k)$. $\square$

We are interested in properties of the polynomial defining the set of points of maximal class in the case that $D_{\mathfrak{g}}$ is the flat distribution of a free symbol $\mathfrak{g}$. Let $\mathbb{C}[u_{r+1}, \ldots, u_n]$ denote the ring of polynomials in the quasi-impulses associated with a local basis of $D_{\mathfrak{g}}$, *i.e.* the ring of polynomial functions that do not vanish on $D_{\mathfrak{g}}^{\perp}(q)$. Let $\mathsf{GL}(V)$ denote the group of automorphisms of a vector space V. Then, the *automorphism group of the symbol* is the subgroup of $\mathsf{GL}(\mathfrak{g})$ defined [31]:

$$G^0(\mathfrak{g}) := \Big\{ g \in \mathsf{GL}(\mathfrak{g}_1) \mid \; g \cdot [X, Y] = [g \cdot X, g \cdot Y] \text{ for all } X, Y \in \mathfrak{g} \\ \text{and } g \cdot \mathfrak{g}_i = \mathfrak{g}_i \text{ for all } i \in [\mu] \Big\}.$$

The group $G^0(\mathfrak{g})$ preserves symmetries of the flat distribution $D_{\mathfrak{g}}$ and, in particular, can be identified with a subgroup of $\mathsf{GL}(\mathfrak{g}_1)$. Thus, the action of $G^0(\mathfrak{g})$ on $\mathfrak{g}_1$ corresponds exactly with changes of basis on the flat distribution $D_{\mathfrak{g}}$ preserving symmetries of the distribution. This action on the generators of the symbol induces a natural action on the quasi-impulses $\{u_i\}_{i=1}^n$ associated with a compatible basis $\{X_i\}_{i=1}^n$ for $\mathfrak{g}$. Let $g \in G^0(\mathfrak{g})$, define $u := (u_{r+1}, \ldots, u_n)$, and let $g \cdot u$ denote the induced action on the nonzero quasi-impulses on $D^{\perp}(q)$. We say a polynomial $f \in \mathbb{C}[u_{r+1}, \dots, u_n]$ is *invariant* if $f(u) = f(g \cdot u)$ and denote the ring of polynomials invariant under the action of a subgroup H of $G^0(\mathfrak{g})$ by $\mathbb{C}[u_{r+1}, \dots, u_n]^H$.

We say a polynomial $f \in \mathbb{C}[u_{r+1}, \dots, u_n]$ is a *relative invariant* of the action of $g \in G$, for some group G, if $f(g \cdot u) = m(g)f(u)$, where $m : G \to \mathbb{C}$ is called a *multiplier function* or *weight function* [26]. Let $G = \mathsf{GL}_r$ denote the group of invertible $r \times r$ matrices and SL_r the special linear subgroup of invertible matrices with determinant equal to 1. Then $g \in \mathsf{GL}_r$ and $m(g) = \det(g)^k$ for some integer k, *i.e.* it is a multiplicative function of the determinant, and the corresponding relative invariants of the GL_r-action are SL_r-invariants.

Below we prove that if $G^0(\mathfrak{g}) = \mathsf{GL}_r$, *i.e.* if the group of symmetries of the flat distribution $D_\mathfrak{g}$ is the general linear group, then the polynomial defining the set of points of maximal class belongs to $\mathbb{C}[u_{r+1}, \dots, u_n]^{\mathsf{SL}_r}$. We use this fact in Sections 3.2 and 3.3 in Chapter 3, where we study the free symbols with degree of nilpotency 4 and 5.

Corollary 2.3.3: If the automorphism group of the symbol $\mathfrak{g}$ with flat distribution $D_\mathfrak{g}$ is GL_r then $\mathcal{R}_{D_\mathfrak{g}}(q)$ is the complement of an algebraic variety defined by a polynomial in $\mathbb{C}[u_{r+1}, \dots, u_n]^{\mathsf{SL}_r}$.

Proof. Following the proof of Lemma 2.1.8, let $a \in \mathbb{C}[u_{r+1}, \dots, u_n]$ denote the polynomial whose variety $\mathcal{V}(a) := \{\lambda \in W_{D_\mathfrak{g}}(q) \mid a(\lambda) = 0\}$ defines the set of points of maximal class: $\mathcal{R}_{D_\mathfrak{g}}(q) = W_{D_\mathfrak{g}}(q) \backslash \mathcal{V}(a)$. The polynomial a is, in particular, a minor of a matrix with polynomial entries. Let $\mathcal{I}(a)$ denote the principal ideal generated by a and $\mathcal{I}(\mathcal{V}(a))$ denote the ideal of polynomials in $\mathbb{C}[u_{r+1}, \dots, u_n]$ vanishing on $\mathcal{V}(a)$. By Hilbert's Nullstullensatz, $\mathcal{I}(\mathcal{V}(a)) = \sqrt{\mathcal{I}(a)}$, the radical of $\mathcal{I}(a)$.

Since $\mathcal{J}^{(0)}(\lambda)$ and $\mathcal{C}_{D_\mathfrak{g}}$ are well-defined independently of any choice of coordinates, points in $\mathcal{V}(a)$ are fixed under changes of coordinates on $D_{\mathfrak{g}}$. It follows that a must be invariant under the action of $G^0(\mathfrak{g})$ up to a scalar multiple; that is, for any $g \in G^0(\mathfrak{g})$, $a(g \cdot u) = m(g)a(u)$ for some function $m : G^0(\mathfrak{g}) \to \mathbb{C}$. Then, m is multiplicative. Since $G^0(\mathfrak{g}) = \mathsf{GL}_r$ and SL_r is its derived subgroup, $m(g)$ is a multiplicative function of the determinant of g and therefore $m(g) = \det(g)^k$ for some integer k. $\square$

In order to prove a distribution is of maximal class at a point it suffices to consider the class of the flat, left-invariant distribution of its symbol at that point. In the following section we develop a basic abstract theory relating the abnormal extremals of a symbol with the abnormal extremals of any quotient of that symbol. We are ultimately interested in using the above Proposition and the following results to develop a strategy to first prove the maximality of class of all flat distributions of *free* graded nilpotent Lie algebras on r generators with degree of nilpotency μ, and then obtain conditions on when its quotients are of maximal class.

2.4 Quotients of constant symbols of maximal class

Let $\mathfrak{f}$ be a finite-dimensional Lie algebra, $\mathfrak{h} \subset \mathfrak{f}$ an ideal and $\mathfrak{g} := \mathfrak{f}/\mathfrak{h}$. Denote the simply-connected Lie groups of $\mathfrak{f}$, $\mathfrak{h}$, and $\mathfrak{g}$ by F, H and G, respectively. In general, we write $M(\mathfrak{m})$ for the simply-connected Lie group of a finite-dimensional Lie algebra $\mathfrak{m}$. Note that H is a closed normal Lie subgroup of F and $G = F/H$. Let $\pi : F \to G$ be the canonical Lie group epimorphism with differential $\pi_* : T_qF \to T_{\pi(q)}G$, for $q \in F$.

The left cosets, qH, of H in F form a foliation of F. Let $D_{\mathfrak{h}}(e) := \mathfrak{h}$ denote the distribution corresponding to this foliation, with $e \in F$ identity. In particular, $T_q qH = D_{\mathfrak{h}}(q) = \operatorname{Ker} \pi_*$. We have the following result.

Lemma 2.4.1: $T^*_{\pi(q)}G \simeq D^{\perp}_{\mathfrak{h}}(q)$ *for all* $q \in F$.

$$
\begin{tikzcd}
T_qF \arrow[r, "\pi_*"] \arrow[d] & T_{\pi(q)}G \\
T_qF/D_{\mathfrak{h}}(q) \arrow[ur, "\overline{\pi_*}"']
\end{tikzcd}
$$

Figure 2.1: Construction of the isomorphism $\overline{\pi_*} : T_qF/D_{\mathfrak{h}}(q) \simeq T_{\pi(q)}(F/H)$.

Proof. The map $\overline{\pi_*} : T_qF/D_{\mathfrak{h}}(q) \to T_{\pi(q)}G$, defined $\overline{\pi_*}(v + D_{\mathfrak{h}}(q)) = \pi_*(v)$ for all $v \in T_qF$, is the unique induced isomorphism such that the diagram in Figure 2.1 commutes. The dual map is an isomorphism $(\overline{\pi_*})^* : T^*_{\pi(q)}G \to (T_qF/D_{\mathfrak{h}}(q))^*$, given by $(\overline{\pi_*})^*(p) = p \circ \pi_*$. Composition with the isomorphism $\psi : (T_qF/D_{\mathfrak{h}})^* \to D_{\mathfrak{h}}^{\perp}$ yields the isomorphism $T^*_{\pi(q)}G \simeq D_{\mathfrak{h}}^{\perp}$. $\square$

The lemma allows us to uniquely identify every $p \in T^*_{\pi(q)}G$ with a $\hat{p} \in D_{\mathfrak{h}}^{\perp}(q)$ satisfying $\hat{p} = p \circ \pi_*$. Using this, construct a surjective bundle projection $\Pi : D_{\mathfrak{h}}^{\perp} \to T^*G$ defined $\Pi(\hat{p}, q) = (p, \pi(q))$ for every $(\hat{p}, q) \in D_{\mathfrak{h}}^{\perp}$, that commutes with the canonical cotangent bundle projections $\pi_F : T^*F \to F$ and $\pi_G : T^*G \to G$. We obtain the pair of commuting diagrams shown in Figure 2.2.

Let $\mathfrak{s} \in \Lambda^1(T^*F)$ and $\bar{\mathfrak{s}} \in \Lambda^1(T^*G)$ denote the tautological Liouville 1-forms of T^*F and T^*G, respectively, such that $\mathfrak{s}_\lambda = \lambda \circ (\pi_F)_*$ and $\bar{\mathfrak{s}}_{\Pi(\lambda)} = \Pi(\lambda) \circ (\pi_G)_*$ for $\lambda = (\hat{p}, q) \in T^*F$ and $\Pi(\lambda) = (p, \pi(q))$. The differentials of these 1-forms are the canonical symplectic 2-forms $\sigma \in \Lambda^2(T^*F)$ and $\overline{\sigma} \in \Lambda^2(T^*G)$, where $\sigma = \mathrm{d}\mathfrak{s}$ and $\overline{\sigma} = \mathrm{d}\bar{\mathfrak{s}}$.

Lemma 2.4.2: $(\Pi^* \bar{\mathfrak{s}})_\lambda = \mathfrak{s}_\lambda$ *and* $(\Pi^* \overline{\sigma})_\lambda = \sigma_\lambda$ *for every* $\lambda \in D_{\mathfrak{h}}^{\perp}$.

Proof. As shown in Figure 2.2, the pushforwards of the bundle projections commute

$$\begin{tikzcd} D_{\mathfrak{h}}^{\perp} \arrow[r, "\Pi"] \arrow[d, "\pi_F"'] & T^*G \arrow[d, "\pi_G"] \\ F \arrow[r, "\pi"'] & G \end{tikzcd} \qquad \begin{tikzcd} T_\lambda D_{\mathfrak{h}}^{\perp} \arrow[r, "\Pi_*"] \arrow[d, "(\pi_F)_*"'] & T_{\Pi(\lambda)}T^*G \arrow[d, "(\pi_G)_*"] \\ T_qF \arrow[r, "\pi_*"'] & T_{\pi(q)}G \end{tikzcd}$$

Figure 2.2: Bundle projections and their pushforwards satisfy commuting diagrams.

such that $(\pi_G)_* \circ \Pi_* = \pi_* \circ (\pi_F)_*$. Applying definitions,

$$\begin{aligned}(\Pi^* \bar{\mathfrak{s}})_\lambda(\hat{v}) = \bar{\mathfrak{s}}_{\Pi(\lambda)}(\Pi_* \hat{v}) &= \langle p,\ (\pi_G)_* \circ \Pi_* \hat{v}\rangle \\ &= \langle p,\ \pi_* \circ (\pi_F)_* \hat{v}\rangle = \langle \hat{p},\ (\pi_F)_* \hat{v}\rangle = \mathfrak{s}_\lambda(\hat{v}).\end{aligned}$$

Taking differentials we find that $\mathrm{d}\mathfrak{s}_\lambda = (\Pi^* \mathrm{d}\bar{\mathfrak{s}})_\lambda$, so the canonical symplectic 2-forms satisfy the same pullback equation. □

Let $\mathfrak{f}^1 = \mathrm{span}\{X_i\}_{i=1}^r$ and assume that $\mathfrak{f}^1 \cap \mathfrak{h} = \emptyset$. Then the symbol $\mathfrak{g} := \mathfrak{f}/\mathfrak{h}$ is generated by the flat, left-invariant (r, n)-distribution $D_\mathfrak{g}$ defined by $D_\mathfrak{g}(e) = \mathfrak{g}^1$, where $\mathfrak{g}^1 = \mathrm{span}\{\overline{X_i}\}_{i=1}^r$ with $\overline{X_i} := \pi_*(X_i)$ and $n = \dim(G)$.

Lemma 2.4.3: If $\gamma_F : [0, T] \to T^*F$ is a segment of an abnormal extremal of $D_\mathfrak{f}$ with endpoints transverse to the left cosets of H then $\Pi(\gamma_F(t))$ is a segment of an abnormal extremal of $D_\mathfrak{g}$.

Proof. We want to show that $\Pi(\gamma_F(t)) \in D_\mathfrak{g}^\perp$ for all $t \in [0, T]$ and $\Pi_*(\dot{\gamma}_F(t)) \in \mathrm{Ker}\,\overline{\sigma}|_{D_\mathfrak{g}^\perp}$ a.e. on $[0, T]$.

Consider some $\overline{\lambda} = (p, \pi(q)) = \Pi(\lambda)$ in T^*G with $\lambda = (\hat{p}, q) \in D_\mathfrak{h}^\perp$ a point in its preimage. For all $i \in [r]$, let u_i be the quasi-impulse of $X_i \in \mathfrak{f}^1$ and v_i the quasi-impulse of $\overline{X_i} = \pi_*(X_i) \in \mathfrak{g}^1$. Then,

$$v_i(\overline{\lambda}) = \langle p,\ (\pi_* X_i)(\pi(q))\rangle = \langle p,\ \pi_*(X_i(q))\rangle = \langle \hat{p},\ X_i(q)\rangle = u_i(\lambda).$$

It follows that $\Pi(D_{\mathfrak{h}}^{\perp} \cap D_{\mathfrak{f}}^{\perp}) = D_{\mathfrak{g}}^{\perp}$.

Define $\widetilde{D_{\mathfrak{h}}^{\perp}} := D_{\mathfrak{h}}^{\perp} \cap D_{\mathfrak{f}}^{\perp}$ and let $\vec{w}$ be the characteristic vector field of the abnormal extremals of $D_{\mathfrak{f}}$. Since $\operatorname{Ker}(\sigma|_{D_{\mathfrak{f}}^{\perp}})_{\lambda} = T_{\lambda} D_{\mathfrak{f}}^{\perp} \cap \operatorname{span}\{\vec{u}_i(\lambda)\}_{i=1}^{r}$, we may write $\vec{w}(\lambda) = \sum_{i=1}^{r} \alpha_i \vec{u}_i(\lambda)$, where the α_i are numbers. If $\lambda = (\hat{p}, q) \in \widetilde{D_{\mathfrak{h}}^{\perp}}$ then, with u_X the quasi-impulse of some $X \in \mathfrak{h}$,

$$\mathrm{d}u_X(\vec{w}(\lambda)) = \sum_{i=1}^{r} \alpha_i u_{[X_i, X]}(\lambda) = \sum_{i=1}^{r} \langle \hat{p}, [X_i, X](q) \rangle = 0,$$

where the first equality follows from equation (2.3) and the last from the fact that $\mathfrak{h}$ is an ideal. It follows that $\vec{w}$ is tangent to $D_{\mathfrak{h}}^{\perp}$, so if the endpoints of γ_F belong to $\widetilde{D_{\mathfrak{h}}^{\perp}}$ then so does the entire segment.

Define $\widetilde{\sigma} := \sigma|_{\widetilde{D_{\mathfrak{h}}^{\perp}}}$ and let $\lambda = (\hat{p}, q) \in \widetilde{D_{\mathfrak{h}}^{\perp}}$, $\overline{\lambda} = \Pi(\lambda)$, $\hat{v} \in \operatorname{Ker} \widetilde{\sigma}_{\lambda}$, $v = \Pi_*(\hat{v})$ and $w \in T_{\overline{\lambda}} T^*G$. Since Π is surjective there is a $\hat{w} \in T_{\lambda} D_{\mathfrak{h}}^{\perp}$ such that $\Pi_*(\hat{w}) = w$. Then, using the preceding results and Lemma 2.4.2, we have that

$$(\overline{\sigma}|_{D_{\mathfrak{g}}^{\perp}})_{\overline{\lambda}}(v, w) = (\Pi^* \overline{\sigma})_{\lambda}(\hat{v}, \hat{w}) = \widetilde{\sigma}_{\lambda}(\hat{v}, \hat{w}).$$

Thus, if $\hat{v} \in \operatorname{Ker} \widetilde{\sigma}$ then $v = \Pi_*(\hat{v}) \in \operatorname{Ker} \overline{\sigma}|_{D_{\mathfrak{g}}^{\perp}}$, which completes the proof. $\square$

We have shown that the abnormal extremals of $D_{\mathfrak{g}}$, which evolve in $D_{\mathfrak{g}}^{\perp} \subset T^*G$, are exactly the abnormal extremals of $D_{\mathfrak{f}}$ that are everywhere transverse to the left cosets qH. Recall that $\mathcal{R}_{D_{\mathfrak{f}}}(q)$ denotes the subset of T_q^*M at which $D_{\mathfrak{f}}$ is of maximal class at $q \in F$. It remains to determine when $\mathcal{R}_{D_{\mathfrak{f}}}(q) \neq \emptyset$ implies $\mathcal{R}_{D_{\mathfrak{g}}}(\pi(q)) \neq \emptyset$.

3. RANK-2 DISTRIBUTIONS OF MAXIMAL CLASS

3.1 Properties of $(2, n)$-distributions

In this section we review basic properties of the abnormal extremals of $(2, n)$-distributions, review known results for $n \leq 6$ from [30] and then discuss the maximality of class of $(2, n)$-distributions with $n \geq 6$.

It was shown in [29] that the restriction of the canonical symplectic 2-form σ is degenerate only on a codimension-1 submanifold of $D^{\perp}$.

Lemma 3.1.1: If D is a $(2, n)$-distribution then $\operatorname{Ker} \sigma|_{D^{\perp}}(\lambda) = H_D(\lambda) \subset T_\lambda D^{\perp}$ *if and only if $\lambda \in (D^2)^{\perp}$, and is otherwise trivial.*

It follows from this and Definitions 2.1.1 and 2.1.3 that abnormal extremals of rank-2 distributions evolve in the codimension-1 submanifold $(D^2)^{\perp} \subset D^{\perp}$. These facts have some additional consequences.

Lemma 3.1.2: An absolutely continuous curve $\gamma : [0, T] \to T^*M$ is an abnormal extremal of a $(2, n)$-distribution D if and only if

(A1') $\gamma(t) \in (D^2)^{\perp}$ for all $t \in [0, T]$ and

(A2') $\dot{\gamma} \in H_D(\lambda)$ a.e. in $[0, T]$.

Moreover, γ is a regular abnormal extremal of D if and only if

(A3') $\gamma(t) \in W_D = (D^2)^{\perp} \backslash (D^3)^{\perp}$ for all $t \in [0, T]$ and

(A4') $\dot{\gamma} \in \mathcal{C}_D = T_\lambda (D^2)^{\perp} \cap H_D(\lambda)$ a.e. in $[0, T]$.

For a proof, see Section 2.1 of [29], in particular Proposition 2.2, Corollary 2.1 and the discussion immediately following.

Let D be a $(2, n)$-distribution with $\dim(D^3) = 5$. Construct a local basis $\{X_1, X_2\}$

for D at $q \in M$ and complete it to a basis $\mathcal{X} = \{X_i\}_{i=1}^n$ for T_qM such that

$$X_3 = [X_1, X_2], \quad X_4 = [X_1, [X_1, X_2]], \quad \text{and} \quad X_5 = [X_2, [X_1, X_2]]. \tag{3.1}$$

Let $\{c_{ji}^k\}_{i,j,k=1}^n$ be the structure functions associated with local basis $\mathcal{X}$ and note in particular that the quasi-impulses $\{u_i\}_{i=1}^n$ of basis $\mathcal{X}$ provide coordinates on T_q^*M, from which we construct coordinates $\{\partial u_i\}_{i=1}^n$ on $T_\lambda(T_{\pi(\lambda)}^*M)$. Recall equation (2.6) for the Hamiltonian lift of u_i in these coordinates:

$$\vec{u}_i = X_i + \sum_{j=1}^n \sum_{k=1}^n c_{ji}^k u_k \partial u_j.$$

We now describe the flag of the lifted distribution and the characteristic line bundle in these coordinates. Recall that a flag of subspaces of some vector space V is said to be *complete* if the growth in dimension between adjacent subspaces of the flag is exactly 1 until the flag saturates the vector space. The following Proposition is based on proofs of similar statements in [30].

Proposition 3.1.3: The characteristic line bundle of a $(2, n)$-distribution D in a local basis $\mathcal{X}$ satisfying (3.1) is equal to

$$\mathcal{C}_D(\lambda) = \text{span}\{u_4\,\vec{u}_2 - u_5\,\vec{u}_1\}(\lambda). \tag{3.2}$$

Moreover, the lift of D in the same coordinates is

$$\mathcal{J}^{(0)}(\lambda) = \text{span}\{\,\vec{u}_1 - u_4\partial u_3,\, \vec{u}_2 - u_5\partial u_3, \partial u_4, \ldots, \partial u_n\}, \tag{3.3}$$

the growth in the flag of $\mathcal{J}^{(0)}$ is $\dim \mathcal{J}^{(i+1)}(\lambda) - \dim \mathcal{J}^{(i)}(\lambda) \leq 1$ for all $i \geq 0$, and the flag is complete if and only if $\dim \mathcal{J}^{(n-3)}(\lambda) = 2n - 4$.

Proof. We lift the distribution over the set $W_D = (D^2)^\perp \backslash (D^3)^\perp$ where we have, by Lemma 3.1.2, that $\mathcal{J}^{(0)}(\lambda) = \Big(T_\lambda(T_{\pi(\lambda)}^*M) + \operatorname{Ker} \sigma|_{D^\perp}(\lambda)\Big) \cap T_\lambda(D^2)^\perp$. Set $V := T_\lambda(T_{\pi(\lambda)}) \cap T_\lambda(D^2)^\perp = \text{span}\{\partial u_j\}_{j=4}^n$ and $U := \text{span}\{\,\vec{u}_1 - u_4\partial u_3,\, \vec{u}_2 - u_5\partial u_3\}$ and observe that $U + V = \mathcal{J}^{(0)}$ is a rank $(n-1)$ distribution in TT^*M. Recall that the flag is contained in the corank-1 distribution Δ, cf. Lemma 2.1.6, and since $\dim(D^2)^\perp(q) = n-3$, $\dim\Delta(\lambda) = 2n-4$ at any $\lambda \in (D^2)^\perp$.

To compute subspace $\mathcal{J}^{(1)}$, take $\vec{h}$ as given in (3.2) and observe that $[\vec{h}, \partial u_i] \in \text{span}\{\partial u_j\}_{j=4}^n$ for all $4 \leq i \leq n$. It follows that $[\vec{h}, U] = \text{span}\{\vec{u}_3 + v\}$, where $v \in \text{span}\{\partial u_i\}_{i=4}^n \subset \mathcal{J}^{(0)}(\lambda)$, and hence $\dim\mathcal{J}^{(1)}(\lambda) - \dim\mathcal{J}^{(0)}(\lambda) = 1$. Therefore, $\dim\mathcal{J}^{(i+1)}(\lambda) - \dim\mathcal{J}^{(i)}(\lambda) \leq 1$ for all $i \leq n-3$. If the growth in the flag is exactly 1 until the flag stabilizes at $\Delta(\lambda)$ then it is complete if and only if $i = 2n-4-(n-1) = n-3$ and $\mathcal{J}^{(i)}(\lambda) = \Delta(\lambda)$ if and only if $\dim\mathcal{J}^{(n-3)}(\lambda) = 2n-4$. □

It follows that any $(2,n)$-distribution is of maximal class at a point $q \in M$ if and only if the lifted distribution has a complete flag at some $\lambda \in W_D(q) = (D^2)^\perp \backslash (D^3)^\perp(q)$. The class of any $(2,n)$-distribution with $n \leq 6$ was addressed in [30] via direct calculations using the observations in the above Proposition. We summarize these results as follows.

Proposition 3.1.4 (Zelenko (2006)): Let D be a $(2,n)$-distribution. If the small growth vector of D is $(2,3,4)$, $(2,3,5)$ or $(2,3,5,6)$ then D is of maximal class.

The case $n = 4$ was proven in Proposition 3.1 of [30] and cases $n = 5, 6$ with $\dim D^3 = 5$ in Propositions 3.5 and 3.6. Note that if $n = 3$ then a $(2,3)$-distribution is a so-called contact distribution and the local equivalence of contact distributions is essentially a consequence of the Darboux Theorem [12]. Moreover, in [17], Doubrov and Zelenko also showed that if D has small growth vector $(2,3,4,\ldots,5)$ then D is either a *Goursat* distribution [25], or appropriate quotients of D yield a $(2,3,5)$ distribution of maximal class.

From Proposition 3.1.4 we conclude that the flag of the lifted distribution $\mathcal{J}^{(0)}$ for any $(2,n)$-distribution with $n \geq 6$ and small growth vector $(2,3,5,\ldots)$ does not stabilize at $\mathcal{J}^{(3)}$. We henceforth restrict our attention to such distributions and prove in our next result that to determine if a $(2,n)$-distribution is maximal class at a point it suffices to compute the determinant of a $n-5 \times n-5$ matrix.

For all $i \in \{1,\ldots,n-5\}$ and $j \in \{6,\ldots,n\}$ define homogeneous polynomials $a_i^j \in \mathbb{C}[u_4,\ldots,u_n]$ by the iterated Lie brackets

$$(\operatorname{ad}\vec{h})^i(u_4\,\vec{u}_5 - u_5\,\vec{u}_4) = \sum_{k=1}^{5} a_i^k\,\vec{u}_k + \sum_{j=6}^{n} a_i^j\,\vec{u}_j, \tag{3.4}$$

and construct an $n-5 \times n-5$ matrix $\mathcal{A}$ with entries given by these polynomials, such that

$$\mathcal{A} = \begin{pmatrix} a_1^6 & \cdots & a_1^n \\ \vdots & \ddots & \vdots \\ a_1^n & \cdots & a_{n-5}^n \end{pmatrix} \tag{3.5}$$

We call this the *matrix associated with the flag* because, as we show in the proof of the following Proposition, it describes the growth in the flag of the lifted distribution. The i^{th} row of $\mathcal{A}$ contains the polynomial coefficients of the quasi-impulses $\vec{u}_6,\ldots,\vec{u}_n$ in the i^{th} iterated Lie bracket of the characteristic vector field $\vec{h} \in \mathcal{C}_D$ with the vector field $u_4\,\vec{u}_5 - u_5\,\vec{u}_4$. Each polynomial a_i^j is homogeneous of total degree $i+1$.

Proposition 3.1.5: *If D is a $(2,n)$-distribution with small growth vector $(2,3,5,\ldots)$ then it is of maximal class $n-3$ at some $q \in M$ if and only if $\det(\mathcal{A}(\lambda)) \not\equiv 0$ for some $\lambda \in W_D(q)$.*

Proof. Recall from the proof of Proposition 3.1.3 that $\mathcal{J}^{(1)} \bmod \mathcal{J}^{(0)} = \operatorname{span}\{\vec{u}_3 + v\}$, where $v \in \operatorname{span}\{\partial u_i\}_{i=4}^n \subset \mathcal{J}^{(0)}(\lambda)$ is some vector that does not affect the growth in the flag. This implies $\mathcal{J}^{(2)} \bmod \mathcal{J}^{(1)} = \operatorname{span}\{[\vec{h},\vec{u}_3]\} = \operatorname{span}\{u_4\,\vec{u}_5 - u_5\,\vec{u}_4\}$ modulo terms in $\{\partial u_i\}_{i=4}^n$. It follows that the flag is complete at $\lambda \in W_D(q)$ if and only if the $n-5$ vectors $\left\{(\operatorname{ad}\vec{h})^i(u_4\,\vec{u}_5 - u_5\,\vec{u}_4)\right\}_{i=1}^{n-5}(\lambda)$ are linearly independent.

However, note that it suffices to take Lie brackets modulo $\operatorname{span}\{\vec{u}_i\}_{i=1}^3$ since

$\mathrm{span}\{\vec{u}_i\}_{i=1}^{3}(\lambda) \subset \mathcal{J}^{(1)}(\lambda)$. Moreover, it suffices to only consider the growth of the flag in a basis consisting of linear combinations of the vector fields $\{\vec{u}_j\}_{j=6}^{n}$ alone, since $\pi_* \,\mathrm{span}\{\vec{u}_i\}_{i=1}^{5}(\lambda) = D^3(\pi(\lambda))$, where $\dim D^3(q) = 5$ for all $q \in M$. $\square$

We have shown that for $(2, n)$-distributions D the polynomial of Lemma 2.1.8 defining the set of points where the flag of the lifted distribution is complete is $\det(\mathcal{A}) \in \mathbb{C}[u_4, \dots, u_n]^{\mathsf{SL}_2}$. In the following sections we show that all distributions with small growth vector $(2, 3, 5, 7$ or $8)$ are of maximal class, and all distributions with small growth vector $(2, 3, 5, 8, 14)$ are of maximal class. Our proofs rely on calculations that make use of elementary invariant theory and representation theory.

3.2 The distribution of the free step 4 symbol is maximal class

Let $\mathfrak{g} = \oplus_{i=1}^{4}\mathfrak{g}_i$ denote the free, fundamental symbol with 2 generators and step, or degree of nilpotency, equal to 4. Let $D_{\mathfrak{g}}$ denote its flat, left-invariant distribution and fix a Hall basis $\mathcal{X} = \{X_i\}_{i=1}^{n}$ for the Lie algebra consistent with (3.1):

$$\begin{aligned} &&& X_3 = [X_1, X_2], \\ && X_4 = [X_1, X_3], \quad & X_5 = [X_2, X_3], \\ & X_6 = [X_1, X_4], \quad & X_7 = [X_2, X_4], \quad & X_8 = [X_2, X_5]. \end{aligned} \tag{3.6}$$

Each subspace $\mathfrak{g}_i$ consists of brackets of length i with the above elements as basis: $\mathfrak{g}_1 = \mathrm{span}\{X_1, X_2\}$, $\mathfrak{g}_2 = \mathrm{span}\{X_3\}$, $\mathfrak{g}_3 = \mathrm{span}\{X_4, X_5\}$, and $\mathfrak{g}_4 = \mathrm{span}\{X_6, X_7, X_8\}$.

The matrix $\mathcal{A}$ of Proposition 3.1.5 is constructed as follows. Compute the iterated

Lie brackets of $\vec{h} = u_4\,\vec{u}_2 - u_5\,\vec{u}_1$ with the vector field $u_4\,\vec{u}_5 - u_5\,\vec{u}_4$:

$$\begin{aligned}
(\mathrm{ad}\,\vec{h})(u_4\,\vec{u}_5 - u_5\,\vec{u}_4) &= -\vec{h}(u_5)\,\vec{u}_4 + \vec{h}(u_4)\,\vec{u}_5 + u_5^2\,\vec{u}_6 - 2u_4u_5\,\vec{u}_7 + u_4^2\,\vec{u}_8,\\
(\mathrm{ad}\,\vec{h})^2(u_4\,\vec{u}_5 - u_5\,\vec{u}_4) &= -\vec{h}^2(u_5)\,\vec{u}_4 + \vec{h}^2(u_4)\,\vec{u}_5 + 3u_5\,\vec{h}(u_5)\,\vec{u}_6\\
&\quad - 3\,\vec{h}(u_4u_5)\,\vec{u}_7 + 3u_4\,\vec{h}(u_4)\,\vec{u}_8,\\
(\mathrm{ad}\,\vec{h})^3(u_4\,\vec{u}_5 - u_5\,\vec{u}_4) &= -\vec{h}^3(u_5)\,\vec{u}_4 + \vec{h}^3(u_4)\,\vec{u}_5 + \Big(3\,\vec{h}(u_5)^2 + 5u_5\,\vec{h}^2(u_5)\Big)\,\vec{u}_6\\
&\quad - \Big(5\,\vec{h}^2(u_4)u_5 + 6\,\vec{h}(u_4)\,\vec{h}(u_5) + 5u_4\,\vec{h}^2(u_5)\Big)\,\vec{u}_7\\
&\quad + \Big(3\,\vec{h}(u_4)^2 + 5u_4\,\vec{h}^2(u_4)\Big)\,\vec{u}_8.
\end{aligned}$$

Define the polynomial $p_1 := u_4\,\vec{h}(u_5) - \vec{h}(u_4)u_5 = u_4^2u_8 - 2u_4u_5u_7 + u_5^2u_6$. Note that p_1 has the form of a generalized Wronskian of the functions u_4 and u_5 and that $\vec{h}(p_1) = 0$. Then, by Proposition 3.1.5, the flag of the lift of $D_{\mathfrak{g}}$ is complete if and only if the determinant of the matrix

$$\mathcal{A} = \begin{pmatrix} u_5^2 & -2u_4u_5 & u_4^2 \\ \frac{3}{2}\,\vec{h}(u_5^2) & -3\,\vec{h}(u_4u_5) & \frac{3}{2}\,\vec{h}(u_4^2) \\ 2\,\vec{h}^2(u_5^2) - \vec{h}(u_5)^2 & 2\,\vec{h}(u_4)\,\vec{h}(u_5) - 4\,\vec{h}^2(u_4u_5) & 2\,\vec{h}^2(u_4^2) - \vec{h}(u_4)^2 \end{pmatrix} \tag{3.7}$$

is nonzero. It is straightforward to show that $\det(\mathcal{A}) = 9p_1^3$, which immediately implies that the flat distribution of the free step 4 symbol is everywhere of maximal class.

Recall that although we fixed a choice of local basis (3.6), the flag of the lifted distribution is well-defined independently of any choice of basis. We proved in Corollary 2.3.3 that the set of points of maximal class of the flat distribution of a free symbol is defined by a polynomial invariant under the action of the special linear group. It follows that the polynomial p_1 must be an SL_2-invariant, with the action induced by changes of coordinates on $\mathfrak{g}_1$, described as follows: let $\mathbb{C}[u_4, \dots, u_{14}]$ denote the polynomial ring of quasi-impulses and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathsf{SL}_2$. Changes of coordinates on

$\mathfrak{g}_1 = \operatorname{span}\{X_1, X_2\}$ transform basis elements X_1, X_2 in the Hall basis $\mathcal{X}$ into basis elements $\widetilde{X_1} = aX_1 + bX_2$ and $\widetilde{X_2} = cX_1 + dX_2$. Setting $u = (u_4, \dots, u_8)$, the induced action on the ring of quasi-impulses can be described as the following matrix vector product:

$$g \cdot u = \begin{pmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & a^2 & 2ab & b^2 \\ 0 & 0 & ac & ad+bc & bd \\ 0 & 0 & c^2 & 2cd & d^2 \end{pmatrix} \begin{pmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} =: A(g)\, u$$

A polynomial $f \in \mathbb{C}[u_4, \dots, u_8]$ is invariant if and only if $f(u) = f(A(g)u)$ for every $g \in \mathsf{SL}_2$ and we denote the ring of polynomials invariant under this action by $\mathbb{C}[u_4, \dots, u_8]^{\mathsf{SL}_2}$. This ring of invariants is finitely generated (c.f. Theorem 2.2.10 of [14]) and it is straightforward to calculate the generators of this ring using an implementation of Derksen's Algorithm [14, Section 4.1.1].

Using the `rinvar` library [10] of `Singular` [13], we find that the invariant ring $\mathbb{C}[u_4, \dots, u_8]^{\mathsf{SL}_2}$ has exactly two generators: $p_1 = u_4^2 u_8 - 2u_4 u_5 u_7 + u_5^2 u_6$ and its discriminant $p_0 := u_7^2 - u_6 u_8$. In particular, since every polynomial entry in the i^{th} row of matrix (3.7) is a homogeneous polynomial of total degree $i+1$ it is straightforward to show that the determinant is a homogeneous polynomial of total degree 27 with every monomial of degree 6 in u_4 and u_5 and degree 3 in u_6, u_7, u_8, without calculating the determinant. Since, by Corollary 2.3.3, the determinant is a $\mathfrak{sl}_2$-invariant we conclude that $\det(\mathcal{A}) = a\, p_1^3$, where $a \in \mathbb{C}$ is some constant and that the set of points of maximal class is $W_D(q) \backslash \mathcal{V}(p_1)$.

Thus, by knowing the generators of the invariant ring and the degrees of the homogeneous polynomial entries of the matrix associated with the flag we were able to determine the set of points of maximal class without taking any determinants. It will be important to develop approaches to finding the set of points of maximal class without directly computing the determinant of a $n-5 \times n-5$ matrix for the free step 5, and higher, symbols. In particular, improvements on the implementation of Derksen's algorithm in [10] will be necessary in order to compute the invariant generators of the ring of invariants in the case of the free step 5 symbol. In the following section we use representation theory to determine the set of points of maximal class without computing determinants.

In Section 2.3 we proved that in order to determine the maximal class of all $(2,n)$-distributions it suffices to reduce our analysis to all flat distributions of symbols of $(2,n)$-distributions. This suggested a strategy for first proving that a free symbol has a flat distribution of maximal class and then proving that all symbols obtained as quotients of this free symbol have flat distributions of maximal class. We explored aspects of this idea in Section 2.4. In the Proposition above we showed that the flat distribution of the free step 4 symbol is of maximal class and it follows from Proposition 2.3.2 above that all distributions with small growth vector $(2,3,5,8)$ at a point are of maximal class. This leads to the following result.

Corollary 3.2.1: If D has small growth $(2,3,5,7$ or $8)$, then D is everywhere of maximal class.

Proof. If D has small growth vector $(2,3,5,8)$ near a point $q \in M$ then its symbol $\mathfrak{g}$ at q is isomorphic to the free step 4 symbol, whose flat, left-invariant distribution is maximal class. In this case we calculated the determinant of matrix (3.7) associated with the flag of the lift of the flat distribution, proving that the set of points of maximal class is the complement of the algebraic variety $\mathcal{V}(p_1)$, where $p_1 = u_4\vec{h}(u_5) - \vec{h}(u_4)u_5 = u_4^2u_8 - 2u_4u_5u_7 + u_5^2u_6$ is an invariant generator of the ring of SL_2-invariants.

If D is a distribution with small growth vector $(2,3,5,7)$ then its symbol at a point with small growth vector $(2, 3, 5, 7)$ is obtained as a quotient of the free step 4 symbol by a 1-dimensional subspace defined by a linear combination of the vectors X_6, X_7 and X_8. It follows that the polynomial defining the points of maximal class in the quotient symbol is obtained as the restriction of p_1 to the corresponding hyperplane. However, p_1 has no linear factor (it is irreducible) and therefore this restriction is not identically zero, yielding the desired result. $\square$

In the next section we consider the class of the flat distribution of the free step 5 symbol, a 14-dimensional Lie algebra. The matrix associated with the flag of its lifted flat distribution is a 9×9 matrix with homogeneous polynomial entries, whose determinant is a degree 54 polynomial which is difficult to expand on standard commercial computer hardware. Since our ultimate goal is to extend our work to free symbols of step 6 and higher, we develop basic tools from elementary representation theory in order to construct the invariant polynomials defining the points where the flag of the lift of the flat, left-invariant distribution of the free step 5 symbol is not complete, without computing the determinant of the matrix associated with the flag.

3.3 The distribution of the free step 5 symbol is maximal class

We reuse the notation of the previous section and take $\mathfrak{g}$ to denote the free step 5 symbol, *i.e.* the graded nilpotent Lie algebra on 2 generators with degree of nilpotency 5. The dimension of this Lie algebra is $n = 14$. Fix a Hall basis $\mathcal{X} = \{X_i\}_{i=1}^{14}$ consisting of the following elements in addition to those defined in (3.1) and (3.6):

$$\begin{aligned} X_9 &= [X_1, X_6], \quad X_{10} = [X_2, X_6], \quad X_{11} = [X_2, X_7], \\ X_{12} &= [X_2, X_8], \quad X_{13} = [X_3, X_4], \quad X_{14} = [X_3, X_5]. \end{aligned} \tag{3.8}$$

Since every entry of the matrix (3.5) is generated by iterated Lie brackets of vector fields with homogeneous polynomial coefficients and every entry in the i^{th} row of $\mathcal{A}$ consists of homogeneous polynomials of total degree $i+1$, $\det(\mathcal{A})$ is a homogeneous polynomial of total degree 54. Although it is difficult to evaluate this determinant directly, it is straightforward to determine the *multi-degree* of every monomial in it, as follows.

First, recall that the *coordinate ring* $k[V]$ of a vector space V over a field k is the polynomial ring of regular functions on it, *i.e.* those polynomials whose restriction to V are not identically zero. It follows that the coordinate ring $\mathbb{C}[\mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_5] \simeq \mathbb{C}[u_4, \ldots, u_{14}]$.

We say that a polynomial $f \in \mathbb{C}[\mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_5]$ is *multi-homogeneous* of *multi-degree* (n_3, n_4, n_5) if every monomial in f is a sum of monomials of total degree n_i in the quasi-impulses corresponding to $\mathfrak{g}_i$, $i = 3$, 4 or 5. Therefore, the space of multi-homogeneous polynomials in $\mathbb{C}[\mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_5]$ of multi-degree (n_3, n_4, n_5) is the vector space

$$W(n_3, n_4, n_5) := \operatorname{Sym}^{n_3} \mathfrak{g}_3 \otimes \operatorname{Sym}^{n_4} \mathfrak{g}_4 \otimes \operatorname{Sym}^{n_5} \mathfrak{g}_5 \tag{3.9}$$

where $\operatorname{Sym}^n V$ denotes the n^{th} symmetric power of a vector space V.

Observe that if f is multi-homogeneous of multi-degree (n_3, n_4, n_5), then the multi-degrees of the polynomial coefficients in the Lie bracket

$$[\vec{h}, f\,\vec{u}_i] = \vec{h}(f)\,\vec{u}_i + u_4 f[\vec{u}_2, \vec{u}_i] - u_5 f[\vec{u}_1, \vec{u}_i] - f\,\vec{u}_i(u_4)\,\vec{u}_2 + f\,\vec{u}_i(u_5)\,\vec{u}_1$$

are easily obtained from the multi-degree of f, yielding recursive rules for calculating all possible multi-degrees in every polynomial entry of the matrix $\mathcal{A}$ associated with the flag of the lifted distribution, assuming that no cancellations occur. Using this fact, we predict that $\det(\mathcal{A})$ is the sum of at most 15 different multi-homogeneous invariants of multi-degrees (38, 2, 14), ..., (26, 26, 2), (25, 28, 1), (24, 30, 0).

We are thus able to calculate properties of the determinant without calculating the determinant directly.

Define the set $U := \{\lambda \in W_D(q) \mid u_4(\lambda) = 0 \text{ and } u_5(\lambda) \neq 0\}$. Evaluating the matrix associated with the flag at all points $\lambda \in U$, we find that the determinant is the sum of 13 invariant polynomials of multi-degree $(38, 2, 14)$, ..., $(26, 26, 2)$:

$$\det(\mathcal{A}|_U) = \sum_{n_3=26}^{38} u_5^{n_3} f_{n_3} \tag{3.10}$$

where each term in the summand is the restriction of a multi-homogeneous invariant in the determinant to the set U, indexed by its total degree in the terms u_4, u_5. For example, $u_5^{38} f_{38}$ is restriction of the multi-homogeneous invariant of multi-degree $(38, 2, 14)$. We have proven the following result.

Proposition 3.3.1: The flat left-invariant distribution $D_{\mathfrak{g}}$ of the free step 5 symbol $\mathfrak{g}$ is of maximal class at all $q \in M(\mathfrak{g})$.

We devote the remainder of this section to the description of the elementary representation theory necessary for reconstructing $\det(\mathcal{A})$ from $\det(\mathcal{A}|_U)$. First, recall that a polynomial is invariant under a Lie algebra action if and only if it is invariant under the corresponding action of its simply-connected Lie group; see Lecture 8 [19], for example, for full details. Recall that the matrices $\mathsf{x} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\mathsf{y} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $\mathsf{h} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ form a basis for $\mathfrak{sl}_2$. The $\mathfrak{sl}_2$-action induced on every $f \in \mathbb{C}[u_4, \ldots, u_{14}]$ by changes of coordinates on $\mathfrak{g}_1$ defines vector fields $\mathsf{x} \cdot f =: v_{\mathsf{x}}(f)$, $\mathsf{y} \cdot f =: v_{\mathsf{y}}(f)$ and

$\mathsf{h}\cdot f =: v_{\mathsf{h}}(f)$, where

$$\begin{aligned}
v_{\mathsf{x}} &= u_5\frac{\partial}{\partial u_4} + 2u_7\frac{\partial}{\partial u_6} + u_8\frac{\partial}{\partial u_7} + (3u_{10}+2u_{13})\frac{\partial}{\partial u_9} + 2u_{11}\frac{\partial}{\partial u_{10}} \\
&\quad + u_{12}\frac{\partial}{\partial u_{11}} + u_{14}\frac{\partial}{\partial u_{13}}, \\
v_{\mathsf{y}} &= u_4\frac{\partial}{\partial u_5} + u_6\frac{\partial}{\partial u_7} + 2u_7\frac{\partial}{\partial u_8} + u_9\frac{\partial}{\partial u_{10}} + 2u_{10} + u_{13}\frac{\partial}{\partial u_{11}} \\
&\quad + (3u_{11}+u_{14})\frac{\partial}{\partial u_{12}} + u_{13}\frac{\partial}{\partial u_{14}}, \\
v_{\mathsf{h}} &= -u_4\frac{\partial}{\partial u_4} + u_5\frac{\partial}{\partial u_5} + -2u_6\frac{\partial}{\partial u_6} + 2u_8\frac{\partial}{\partial u_8} + -3u_9\frac{\partial}{\partial u_9} \\
&\quad - u_{10}\frac{\partial}{\partial u_{10}} + u_{11}\frac{\partial}{\partial u_{11}} + 3u_{12}\frac{\partial}{\partial u_{12}} - u_{13}\frac{\partial}{\partial u_{13}} + u_{14}\frac{\partial}{\partial u_{14}}.
\end{aligned}$$

We say that f is an $\mathfrak{sl}_2$-invariant polynomial if and only if $\mathsf{x}\cdot f = \mathsf{y}\cdot f = \mathsf{h}\cdot f = 0$ and denote the ring of invariant polynomials by $\mathbb{C}[u_4,\ldots,u_{14}]^{\mathfrak{sl}_2}$.

A representation is said to be *irreducible* if it has no proper subrepresentation and otherwise *reducible*. Thus, while $\operatorname{Sym}^{n_3}\mathfrak{g}_3$ is an irreducible $\mathfrak{sl}_2$-representation, $\operatorname{Sym}^{n_4}\mathfrak{g}_4 \simeq \operatorname{Sym}^{n_4}(\operatorname{Sym}^2\mathfrak{g}_3)$ and $\operatorname{Sym}^{n_5}\mathfrak{g}_5 \simeq \operatorname{Sym}^{n_5}(\operatorname{Sym}^3\mathfrak{g}_3 \oplus \mathfrak{g}_3)$ are reducible.

Recall also that the eigenvalues and eigenvectors of the action of h are called, respectively, the *weights* and *weight vectors* of the ($\mathfrak{sl}_2$) representation. The subspace spanned by all vectors of a given weight in an irreducible representation is called the *weight space*. If α is a weight then let V_α denote its weight space. The irreducible $\mathfrak{sl}_2$-representations and weight spaces of the free step 5 symbol in terms of the Hall basis (3.8) are

$$\begin{gathered}
\mathfrak{g}_3 = \langle X_4\rangle \oplus \langle X_5\rangle \simeq \langle X_{13}\rangle \oplus \langle X_{14}\rangle \simeq V_{-1}\oplus V_1, \\
\operatorname{Sym}^2\mathfrak{g}_3 \simeq \langle X_6\rangle \oplus \langle X_7\rangle \oplus \langle X_8\rangle \simeq V_{-2}\oplus V_0\oplus V_2, \\
\operatorname{Sym}^3\mathfrak{g}_3 \simeq \langle X_9\rangle \oplus \langle X_{10}\rangle \oplus \langle X_{11}\rangle \oplus \langle X_{12}\rangle \simeq V_{-3}\oplus V_{-1}\oplus V_1\oplus V_3,
\end{gathered}$$

where we used shorthand $\langle\cdot\rangle = \operatorname{span}\{\cdot\}$. If $b=(b_4,\ldots,b_{14})$ is an exponent vector and $u^b := u_4^{b_4}\cdots u_{14}^{b_{14}}$ the corresponding monomial, then the weight of u^b is the eigenvalue of the action $\mathsf{h} \cdot u^b = v_{\mathsf{h}}(u^b)$. The weight of a polynomial in $\mathbb{C}[u_4, \ldots, u_{14}]$ is defined if its monomials all have the same weight.

The following Proposition is a straightforward consequence of Schur's Lemma and other facts from elementary representation theory proven in Lectures 1, 8 and 11 of [19]. Let $W(n_3, n_4, n_5)^{\mathfrak{sl}_2}$ denote the vector space of multi-homogeneous polynomials of multi-degree (n_3, n_4, n_5) that are invariant under the above $\mathfrak{sl}_2$ action.

Proposition 3.3.2: If $g \in W(n_3, n_4, n_5)^{\mathfrak{sl}_2}$ and $n_3 > 0$ then

$$g = u_4^{n_3}\, f_0 + u_4^{n_3-1}\, u_5\, f_1 + \cdots + u_4\, u_5^{n_3-1} f_{n-1} + u_5^{n_3}\, f_{n_3}, \tag{3.11}$$

where $f_j \in \mathrm{Sym}^{n_4}\, \mathfrak{g}_4 \otimes \mathrm{Sym}^{n_5}\, \mathfrak{g}_5$ is a polynomial of monomials of weight $-(2j - n_3)$ satisfying

$$0 = j\, f_j + (\mathsf{y} \cdot f_{j-1}) \quad \text{for } \ 1 \leq j \leq n_3, \ \text{and}$$

$$0 = (n_3 - j)\, f_j + (\mathsf{x} \cdot f_{j+1}) \quad \text{for } \ 0 \leq j \leq n_3 - 1,$$

and $0 = \mathsf{y} \cdot f_{n_3} = \mathsf{x} \cdot f_0$.

Since basis element y is often called a weight lowering operator and x a weight raising operator, we call f_0 the *highest weight polynomial* of f and f_{n_3} the *lowest weight polynomial*. The Proposition establishes that if either f_0 or f_{n_3} is known, then the entire invariant g can be reconstructed from repeated application of either y or x to f_0 or f_{n_3}, respectively. The polynomials $f_{38}, \ldots, f_{26}$ defined in equation (3.10) are exactly the lowest weight polynomials of the 13 multi-homogeneous invariants in $\det(\mathcal{A})$. Since it is difficult to expand $\det(\mathcal{A})$ in `Mathematica` on standard commercial hardware[1], calculation of the smaller polynomial $\det(\mathcal{A}|_U)$ and then application of the Proposition above allows us to reconstruct the determinant without

[1]All calculations in this thesis were completed on a 64-bit machine with 4th generation Intel Core i5 processors.

taking determinants. Define the invariant polynomial

$$p_2 := \vec{h}(p_1) = -u_5^3 u_9 + u_5^2 u_4 \left(3u_{10} + 2u_{13}\right) - u_5 u_4^2 \left(3u_{11} + u_{14}\right) + u_4^3 u_{12}$$

and let g_i denote the invariant reconstructed from its lowest weight polynomial f_i. Then, $\det(\mathcal{A}(\lambda)) = \sum_{i=26}^{38} g_i$ where each g_i factors into irreducible invariants as

$$\begin{aligned} g_{38} &= p_2^{10}\, h_{38}, \quad g_{37} = p_2^{8}\, h_{37}, & g_{36} &= p_2^{6}\, h_{36}, \quad g_{35} = p_1\, p_2^{4}\, h_{35}, \\ g_{34} &= p_1^{2}\, p_2^{2}\, h_{34}, \quad g_{33} = p_1^{3}\, h_{33}, & g_{32} &= p_1^{4}\, h_{32}, \quad g_{31} = p_1^{5}\, h_{31}, \\ g_{30} &= p_1^{6}\, h_{30}, \quad g_{29} = p_1^{7}\, h_{29}, & g_{28} &= p_0\, p_1^{8}\, h_{28}, \quad g_{27} = p_0^{3}\, p_1^{9}\, h_{27}, \\ g_{26} &= p_0^{5}\, p_1^{11}\, h_{26}, \end{aligned}$$

and the $\{h_i\}_{i=26}^{38}$ are multi-homogeneous invariants of appropriate multi-degree. These are not too large polynomials, with h_{33} the largest irreducible invariant containing 108,320 monomials. We expect that it will help to rewrite polynomials h_i in terms of the invariant generators of the ring $\mathbb{C}[\mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_5]^{\mathfrak{sl}_2}$, in order to determine those quotient symbols of $\mathfrak{g}$ whose flat, left-invariant distributions remain maximal class.

In the following section we provide an alternative approach to proving the maximality of class of certain kinds of distributions. We use, essentially, the binomial theorem to show that the numeric coefficient of the lowest weight polynomials in the determinant of the matrix associated with the flag is nonzero. This approach is made possible by the relatively simple form of the Lie bracket table of the symbols of these distributions.

3.4 Distributions associated with Monge ODEs are of maximal class

Following [9], we consider so-called Monge differential equations in the form

$$y^{(m)} = F(x, y, y', \ldots, y^{(m-1)}, z, z', \ldots, z^{(n)}), \tag{3.12}$$

where $m \leq n$, $y = y(x), z = z(x)$ and $F : \mathbb{R}^{m+n+3} \to \mathbb{R}$ is a smooth function. We construct a distribution associated with this differential equation as follows: consider the $(m+n+2)$-dimensional submanifold of $\mathcal{E} \subset \mathbb{R}^{m+n+3}$ defined

$$\mathcal{E} = \{q = (x_0, y_0, \ldots, y_m, z_0, \ldots, z_n) \in \mathbb{R}^{m+n+3} \mid y_m = F(q)\}.$$

There exists a regular, bracket-generating $(2, m+n+2)$-distribution on $\mathcal{E}$ canonically associated with Monge equation (3.12) and defined

$$D_{m,n}(q) = \operatorname{span}\left\{\frac{\partial}{\partial x} + \left(\sum_{i=1}^{m} y_i \frac{\partial}{\partial y_{i-1}}\right) + \left(\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_{j-1}}\right),\ \frac{\partial}{\partial z_n}\right\},$$

where, again, $y_m = F(q)$. In [9, Proposition 2.3] Anderson & Kruglikov show that the symbol of $D_{m,n}$ at $q \in \mathcal{E}$ is isomorphic to a certain graded nilpotent Lie algebra.

Define a constant fundamental symbol $\mathfrak{m}_{m,n}$ of degree of nilpotency μ, such that for all $1 \leq m \leq n$, $1 \leq i \leq n+1$ and $1 \leq j \leq m+2$, there exist $X_i, X'_j \in \mathfrak{m}_{m,n}$ defined $X_i := (\operatorname{ad} X_1)^{i-1}(X'_1)$ and $X'_j := [X'_1, (\operatorname{ad} X_1)^{j-2}(X'_1)]$ such that

$$\begin{aligned} X_{i+1} = [X_1, X_i], \quad X'_{i+1} = [X_1, X'_i] &= [X'_1, X_i], \\ \text{and } [X'_1, X'_i] &= 0. \end{aligned} \tag{3.13}$$

In particular, if $i > n+1$ then $X_i = 0$ and if $j > m+2$ then $X'_j = 0$.

Anderson & Kruglikov showed that the elements X_i and X'_j form a Hall basis $\mathcal{H}_{m,n}$ for symbol $\mathfrak{m}_{m,n}$, $m \leq n$:

$$\mathcal{H}_{m,n} = \{X_1, X'_1, X_2\} \cup \left\{X_i\right\}_{i=3}^{n+1} \cup \left\{X'_j\right\}_{j=3}^{m+2}.$$

Moreover, they showed the symbol of $D_{m,n}$ at each $q \in \mathcal{E}$ is isomorphic to the same constant symbol $\mathfrak{m}_{m,n}$.

Proposition 3.4.1 (Anderson & Kruglikov (2006)): Let $\mathfrak{g}_{m,n}(q)$ denote the symbol of $D_{m,n}(q)$ at $q \in \mathcal{E}$. Then, $\mathfrak{g}_{m,n}(q) \simeq \mathfrak{m}_{m,n}$ if and only if $\partial^2 F(q)/\partial z_n^2 \neq 0$ for all $q \in \mathcal{E}$.

Let $D_{\mathfrak{m}}$ be the flat, left-invariant distribution on $M(\mathfrak{m}_{m,n})$ defined by $D_{\mathfrak{m}}(e) := \mathfrak{m}^1_{m,n}(e)$. It follows from Proposition 2.3.2 that if $D_{\mathfrak{m}}$ is of maximal class then $D_{m,n}$ is everywhere of maximal class. We henceforth restrict our attention to this constant symbol.

Let $M(\mathfrak{m}_{m,n})$ denote the simply-connected Lie group of the symbol $\mathfrak{m}_{m,n} = \oplus_{i=1}^{\mu}\mathfrak{m}^i_{m,n}$, where μ is the degree of nilpotency, and equal to $\mu = n+1$ if $m < n$ and $\mu = n+2$ if $m = n$. Let u_i, u'_i and $\vec{u}_i$, $\vec{u}'_i$ denote the quasi-impulses and Hamiltonian lifts associated with Hall basis elements $X_i, X'_i \in \mathcal{H}_{m,n}$, respectively. Then the set of regular abnormal extremals of $D_{\mathfrak{m}}$ is $W_{D_{\mathfrak{m}}} := (D^2_{\mathfrak{m}})^{\perp}\backslash(D^3_{\mathfrak{m}})^{\perp}$, where $(D^2_{\mathfrak{m}})^{\perp} = \{\lambda \in T^*M \mid u_1(\lambda) = u'_1(\lambda) = 0\}$ and $(D^3_{\mathfrak{m}})^{\perp} = \{\lambda \in T^*M \mid u_1(\lambda) = u'_1(\lambda) = u_2(\lambda) = 0\}$. Likewise, rewriting the lift of $D_{\mathfrak{m}}$ in this basis, we find that

$$\mathcal{J}^{(0)} = \mathrm{span}\{\,\vec{u}_1 - u_3\partial u_2,\, \vec{u}_2 - u'_3\partial u_2, \partial u_3, \ldots, \partial u_{n+1}, \partial u'_3, \ldots, \partial u'_{m+2}\},$$

and the characteristic vector field of the flat distribution $D_{\mathfrak{m}}$ is

$$\vec{h} = u_3\,\vec{u}'_1 - u'_3\,\vec{u}_1. \tag{3.14}$$

Recall from the proof of Propositions 3.1.3 and 3.1.5 that it suffices to compute the iterated Lie brackets describing the growth in the flag modulo the span of vectors $\{\,\vec{u}_1,\, \vec{u}'_1,\, \vec{u}_2\}$. In general, for functions f, g and vector fields X, Y, any iterated Lie bracket $\mathrm{ad}^i(fX)(gY)$ modulo $\mathrm{span}\{X\}$ satisfies a binomial formula

$$(\mathrm{ad}\, fX)^k(gY) = \sum_{i+j=k}\binom{k}{i}(fX)^i(g)\,(\mathrm{ad}\, fX)^j(Y).$$

Following Propositions 3.1.3 and 3.1.5, we construct the matrix associated with the flag of $\mathcal{J}^{(0)}$ by defining homogeneous polynomials

$$a_{i,j},\, a'_{i,j} \in \mathbb{C}[u_3, \ldots, u_{n+1}, u'_3, \ldots, u_m]$$

of total degree $i+1$ via the iterated Lie brackets

$$(\text{ad}\,\vec{h})^i(u_3\,\vec{u}_3' - u_3'\,\vec{u}_3) = \sum_{j=3}^{n+1} a_{i,j}\,\vec{u}_j + \sum_{j=3}^{m} a_{i,j}'\,\vec{u}_j', \tag{3.15}$$

where all Lie brackets are taken modulo $\text{span}\{\vec{u}_1, \vec{u}_1', \vec{u}_2\}$. Setting $N = m+n+2$, we let $\mathcal{A}_{m,n}$ denote the $N-5 \times N-5$ matrix (3.5) associated with the flag with polynomial entries $a_{i,j}$ and $a_{i,j}'$:

$$\mathcal{A}_{m,n} = \begin{pmatrix} a_{1,4} & \cdots & a_{1,n+1} & a_{1,4}' & \cdots & a_{1,m+2}' \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & a_{n-2,n+1} & \vdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & a_{n-1,4}' & \cdots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{N-5,4} & \cdots & a_{N-5,n+1} & a_{N-5,4}' & \cdots & a_{N-5,m+2}' \end{pmatrix}.$$

Observe that the main diagonal of $\mathcal{A}_{m,n}$ consists of the polynomials

$$(a_{1,4}, \ldots, a_{n-2,n+1}, a_{n-1,4}', \ldots, a_{m+n-3,m+2}').$$

Our objective is to show that $\det(\mathcal{A}_{m,n}) \not\equiv 0$.

Proposition 3.4.2: The flat, left-invariant distribution $D_{\mathfrak{m}}$ of the constant symbol $\mathfrak{m}_{m,n}$ is of maximal class for all $m \le n$ at all $q \in M(\mathfrak{m}_{m,n})$.

Proof. The monomials in $\det(\mathcal{A}_{m,n})$ are of the form

$$\beta\, u_3^{\alpha_3} \cdots u_{n+1}^{\alpha_{n+1}} \cdot (u_3')^{\alpha_3'} \cdots (u_{m+2}')^{\alpha_{m+2}'},$$

where β is an integer coefficient. We show there exists a monomial in the determinant with highest possible degree in u_3' and nonzero integer coefficient.

We begin by determining monomials in each polynomial $a_{i,j}$, $a_{i,j}'$ on the diagonal of $\mathcal{A}_{m,n}$ with highest possible degree in u_3'. Since we take Lie brackets modulo span$\{\vec{u}_1, \vec{u}_1', \vec{u}_2\}$, we find that equation (3.15) can also be written

$$(\mathrm{ad}\,\vec{h})^i(u_3\vec{u}_3' - u_3'\vec{u}_3) = \sum_{j+k=i}\binom{i}{j}\Big(\vec{h}^j(u_3)(\mathrm{ad}\,\vec{h})^k(\vec{u}_3') \\ + \vec{h}^j(-u_3')(\mathrm{ad}\,\vec{h})^k(\vec{u}_3)\Big). \tag{3.16}$$

When $k = i$ and $j = 0$ the summand equals $u_3(\mathrm{ad}\,\vec{h})^i(\vec{u}_3') - u_3'(\mathrm{ad}\,\vec{h})^i(\vec{u}_3)$. We find the monomials with highest degree $i+1$ in u_3', by expanding the iterated Lie bracket $(\mathrm{ad}\,\vec{h})^i(\vec{u}_3)$. Recall equation (3.14) for the characteristic vector field $\vec{h}$ and the Lie bracket table for the Hall basis $\mathcal{H}_{m,n}$ in equation (3.13). In particular, $[\vec{u}_1, \vec{u}_i] = \vec{u}_{i+1}$ and $[\vec{u}_1', \vec{u}_i] = \vec{u}_{i+1}'$. We find that

$$\begin{aligned}
[\vec{h}, \vec{u}_3] &= u_3[\vec{u}_1', \vec{u}_3] - u_3'[\vec{u}_1, \vec{u}_3] = (-u_3')\,\vec{u}_4 + u_3\,\vec{u}_4', \\
(\mathrm{ad}\,\vec{h})^2(\vec{u}_3) &= \vec{h}(-u_3')\,\vec{u}_4 + (-u_3')(u_3[\vec{u}_1', \vec{u}_4] - u_3'[\vec{u}_1, \vec{u}_4]) + \vec{h}(u_3)\,\vec{u}_4' + u_3[\vec{h}, \vec{u}_4'] \\
&= (-u_3')^2\,\vec{u}_5 + \cdots .
\end{aligned}$$

A straightforward induction implies $-u_3'(\mathrm{ad}\,\vec{h})^i(\vec{u}_3) = (-u_3')^{i+1}\,\vec{u}_{i+3} + \cdots$. Consequently, $(-u_3')^{i+1}$ is the uniquely-occurring monomial of highest degree in u_3' in the polynomial entries $a_{i,i+3}$, where $4 \le i+3 \le n+1$, since the Lie bracket relations (3.13) imply $\vec{u}_{i+3} = 0$ for all $i \ge n-1$. That is, the first $n-2$ entries in the main diagonal of $\mathcal{A}_{m,n}$ are of the form $a_{i,i+3} = (-1)^{i+1}\,(u_3')^{i+1} + \cdots$.

We now describe certain uniquely-occurring monomials of highest degree i in u_3' in the remaining $m-1$ diagonal entries $(a'_{n-1,n+2}, \ldots, a'_{n+m-3,m+2})$. These are the polynomials $a'_{i,i-n+5}$ for $n-1 \le i \le m+n-3$. Fix $j = n-2$ so that $k = i-n+2$, and let $n-2 \le i \le m+n-3$. Then, the corresponding summand in the iterated Lie brackets (3.16) is of the form

$$\binom{i}{n-2}\Big(\vec{h}^{n-2}(u_3)(\mathrm{ad}\,\vec{h})^{i-n+2}(\vec{u}_3') + \vec{h}^{n-2}(-u_3')(\mathrm{ad}\,\vec{h})^{i-n+2}(\vec{u}_3)\Big) \tag{3.17}$$

Applying a straight-forward induction as above, we find that $(\mathrm{ad}\,\vec{h})^{i-n+2}(\vec{u}_3') =$

$(-u_3')^{i-n+2}\,\vec{u}'_{i-n+5} + \cdots$, where $(-u_3')^{i-n+2}$ is the uniquely-occurring monomial of highest degree in the Lie bracket. Likewise, we expand $\vec{h}^{n-2}(u_3) = (-u_3')^{n-2}u_{n+1} + \cdots$ in terms of a uniquely-occurring monomial of highest degree i in u_3'. The summand in equation (3.17) reduces to

$$\binom{i}{n-2}\Big((-u_3')^i u_{n+1}\,\vec{u}_{i-n+5} + \cdots\Big),$$

from which it follows that the remaining $m-1$ terms in the diagonal of $\mathcal{A}_{m,n}$ are of the form $a_{i,i+n-5} = \binom{i}{n-2}(-u_3')^i u_{n+1} + \cdots$.

To summarize, we have identified uniquely-occurring monomials with highest degree in u_3' on the main diagonal of $\mathcal{A}_{m,n}$, where the first $n-2$ entries on the diagonal contain monomials $\{(-u_3')^{i+1}\}_{i=1}^{n-2}$ and the last $m-1$ entries contain monomials $\{\binom{i}{n-2}(-u_3')^i u_{n+1}\}_{i=n-1}^{m+n-3}$. The corresponding monomial in $\det(\mathcal{A}_{m,n})$ formed by the product of the diagonal is

$$\left(\prod_{i=1}^{n-2}(-u_3')^{i+1}\right)\left(\prod_{i=n-1}^{m+n-3}\binom{i}{n-2}(-u_3')^i u_{n+1}\right) = \beta\,(u_3')^{\ell}\,u_{n+1}^{m-1},$$

where $\ell = n-1+\frac{1}{2}(N-6)(N-3)$ and the nonzero numeric coefficient is

$$\begin{aligned}\beta &= (-1)^{\ell+1}\frac{(m+n-3)(m+n-4)^2\cdots(n-1)^{m-1}}{(m-1)!(m-2)!\cdots 2!}\\ &= (-1)^{\ell}\prod_{i=1}^{m-1}\frac{(m+n-(i+2))^i}{i!}.\end{aligned}$$

$\square$

4. CONCLUSION

We have proven various results regarding regular, bracket-generating rank-r distributions D with points of maximal class on an n-dimensional manifold M. In the first chapter, we defined the maximal class property as the condition that there exists a complete flag of subspaces in $T_\lambda T^*M$. We proved our first main result, that a distribution is of maximal class at a point if it possesses a corank-1 abnormal trajectory through that point.

We then showed the maximal class property is an algebraic condition described by a polynomial in the quasi-impulses associated with a local basis, and moreover that this polynomial is invariant under changes of local basis on the distribution by $r \times r$ invertible matrices. We then proved that if the flat distribution of the symbol of the distribution at a point is of maximal class then the distribution itself is of maximal class at that point.

This result provides impetus for our investigation into a strategy for finding all (r, n)-distributions of maximal class. In our fourth main result, we showed that abnormal extremals of a symbol with endpoints satisfying certain transversality constraints project onto the abnormal extremals of the flat distributions of its quotient symbols. This provided motivation for our investigation of the class of the flat distributions of free symbols with r generators and degree of nilpotency greater μ; it remains, however, to establish conditions on when the flat distribution of a quotient symbol is of maximal class, given that the original symbol is of maximal class.

In the previous chapter we showed that the free symbols with 2 generators and degree of nilpotency equal to 4 and 5 are of maximal class, and calculated the invariant polynomials defining the set of points of non-maximal class. This lead to proofs that distributions with small growth vectors $(2, 3, 5, 7 \text{ or } 8)$ or $(2, 3, 5, 8, 14)$ are of maximal class. Then, as an illustration of an alternative strategy to determining the class of certain kinds of $(2, n)$-distributions, we showed via direct calculations that the distributions associated with Monge ordinary differential equations are of maximal class, relying on the simple Lie bracket structure of the symbols of these distributions.

Future work will focus on developing tools for proving the maximality of class of the flat rank-2 distributions of symbols with degree of nilpotency 6 and higher, and to developing new tools for determining when quotient symbols are of maximal class.

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Let $P_t : M \to M$ denote the flow of a non-autonomous vector field $V_t \in \mathsf{Vec}(M)$. It is given, in the chronological calculus, by the formula

$$P_t = \overrightarrow{\exp} \int_0^t V_\tau \, \mathrm{d}\tau.$$

The flow is a diffeomorphism, and the differential of its inverse $(P_t^{-1})_*$ has special notation in the chronological calculus,

$$(P_t^{-1})_* = \operatorname{Ad} \overrightarrow{\exp} \int_0^t V_\tau \, \mathrm{d}\tau.$$

Either expression admits an asymptotic Volterra series expansion, convergence of which is addressed in [3]. Define the simplex $\Delta_n(t) = \{(\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \mid 0 \leq \tau_n \leq \cdots \leq \tau_1 \leq t\}$. The asymptotic series expansion of the flow of non-autonomous vector field V_t is

$$\overrightarrow{\exp} \int_0^t V_\tau \, \mathrm{d}\tau \approx I + \sum_{n=1}^{\infty} \int \cdots \int_{\Delta_n(t)} V_{\tau_n} \circ \cdots \circ V_{\tau_1} \, \mathrm{d}\tau_n \cdots \mathrm{d}\tau_1. \tag{A.1}$$

The expansion of the differential of the inverse flow is similar:

$$\begin{aligned} \operatorname{Ad} \overrightarrow{\exp} \int_0^t V_\tau \, \mathrm{d}\tau &= I + \int_0^t \operatorname{ad} V_\tau \, \mathrm{d}\tau + \int_0^t \int_0^\tau \operatorname{ad} V_{\tau_2} \circ \operatorname{ad} V_{\tau_1} \, \mathrm{d}\tau_1 \, \mathrm{d}\tau + \cdots \\ &= I + \sum_{n=0}^{\infty} \int \cdots \int_{\Delta_n(t)} \operatorname{ad} V_{\tau_n} \circ \cdots \circ \operatorname{ad} V_{\tau_1} \, \mathrm{d}\tau_n \cdots \mathrm{d}\tau_1. \end{aligned}$$

If $V_t = V$ is an autonomous vector field then the flow is written $P_t = \overrightarrow{\exp} \int_0^t V \, \mathrm{d}\tau =: e^{Vt}$ and we write the above expansion more compactly as

$$\operatorname{Ad} e^{Vt} = I + \sum_{n=1}^{\infty} \frac{t^n}{n!} (\operatorname{ad} V)^n. \tag{A.2}$$

We also will require the first *generalized variational formula*, equation (2.4) of [6], for the flow of a vector field subject to an additive perturbation. Let $\hat{X}_\tau$ be a perturbation vector field and X_τ a vector field. Then the generalized variational formula for the flow of $\hat{X}_\tau + X_\tau$ is

$$\overrightarrow{\exp}\int_0^t (\hat{X}_\tau + X_\tau)\,\mathrm{d}\tau = \overrightarrow{\exp}\int_0^t \hat{X}_\tau\,\mathrm{d}\tau \circ \overrightarrow{\exp}\int_0^t (\mathrm{Ad}\ \overrightarrow{\exp}\int_t^\tau \hat{X}_\theta\,\mathrm{d}\theta)\cdot(X_\tau)\ \mathrm{d}\tau \tag{A.3}$$

We also have the following operator differential equation for the differential of the inverse of the flow of a vector field X_τ is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{Ad}\,\overrightarrow{\exp}\int_0^\tau X_\theta\,\mathrm{d}\theta)\cdot(Y) = (\mathrm{Ad}\,\overrightarrow{\exp}\int_0^\tau X_\theta\,\mathrm{d}\theta)\cdot((\mathrm{ad}\,X_\tau)(Y)).$$

The solution of this operator differential equation satisfies

$$(\mathrm{Ad}\,\overrightarrow{\exp}\int_0^\tau X_\theta\,\mathrm{d}\theta)\cdot(Y) = (\overrightarrow{\exp}\int_0^\tau \mathrm{ad}\,X_\theta\,\mathrm{d}\theta)\cdot(Y) \tag{A.4}$$

and if $X_t = X$ is autonomous, then

$$\mathrm{Ad}\,e^{\tau X} = e^{\tau\,\mathrm{ad}\,X}. \tag{A.5}$$